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DECENTRALIZED CONTROL OF LARGE-SCALE SYSTEMS: FIXED MODES, SENSITIVITY AND PARAMETRIC ROBUSTNESS

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FOREWORD

The present work is the result of three years of research with the "Decomposition, Control of Interconnected Systems (D.C.S.I.)" team at the "Automatic and System Analysis Laboratory (L.A.A.S)" of the National Center for Scientific Research (C.N.R.S.) under the direction of A. Costes and D. Esteve. I wish to thank them for welcoming me to the Laboratory.

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The fruit of this collaboration corresponds to the results given in sections VI.4.3. (TRA-85a); VII.2.3b and VII.3.3b (TAR-85c); VII.2.4a, and to publications (TIT-83b) (TRA-84a) (TRA-85b), as well as to the preparation of a book on the subject discussed, containing our joint and personal results (TIT-86).

I wish to thank all of my colleagues of the D.C.S.I. (L.A.A.S.) team, for the numerous discussions we had, for their availability and for the friendly atmosphere they created.

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DECENTRALIZED CONTROL OF LARGE-SCALE SYSTEMS: FIXED MODES, SENSITIVITY AND PARAMETRIC ROBUSTNESS

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CHAPTER 1 - GENERAL DESCRIPTION OF COMPLEX SYSTEMS /1*

I.1 NOTION OF COMPLEXITY

The increasing complexity of dynamic systems has led to a considerably development of mathematical tools for the analysis, optimization and control of such systems.

First, let us try to define this notion of complexity: Simon (SIM-62) defines a complex system as consisting of a large number of parts which are interconnected in a complicated (unsimple) manner. Siljak (SIL-83) introduces the three following characteristics:

a) Large dimension: this aspect is often considered to be the largest source of complexity.

b) Uncertainty: it is assumed that the nature of a complex /2
system cannot be accurately known, whether it be a deterministic or stochastic environment. Uncertainty is essentially found at the interconnection level between the various parts of the system (subsystems). The intrinsic features of a subsystem may be measured or locally predicted in a satisfactory manner in most practical situations.

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*Numbers in the margin indicate pagination in the foreign text.

c) Structural constraints: the existence of structural constraints on the flow of information between subsystems makes the application of conventional control methods difficult, even on small scale systems.

Titli (TIT-83a) makes a distinction between a large system and a complex system. The notion of complexity and complex systems can therefore not be characterized simply and directly, but regardless of the nuance retained, the complexity of the process always remains one of the main challenges to systems theory.

In the text that follows, we shall consider large scale systems which we will call complex systems, of which there are many examples: electrical power networks, urban traffic networks, communication networks, ecological systems, social systems, polytechnic systems, etc.

I.2 METHODS OF CONTROLLING COMPLEX SYSTEMS

Since the mathematical controls which described systems behavior are of considerable size, conventional methods and existing numerical techniques have proven to be inadequate for treating certain problems, particularly because of the presence of:

-Massive calculations and therefore of extensive computer time and high computer costs,

-Large storage requirements,

-Presence of several dynamics in the system,

-Multiple criteria,

-Structural constraints on the flow of information between the different parts of the system, leading to prohibitive computer costs and equipment.

Accordingly, new methods were developed which may be classified: decomposition methods, dimension reduction methods and multi-criteria optimization methods.

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I.3 DECOMPOSITION METHODS

These methods are essentially classed into 3 groups:

I.3.1. Horizontal or Spatial Decomposition

In order to decrease the calculation problems and after defining a coupling system, the process is broken down. This makes it possible to formulate the overall problem based on a certain number of distinct small scale problems, which may therefore be processed in a reasonable amount of time. Each subproblem is therefore the responsibility of a local control unit whose actions are coordinated (hierarchical control) or distributed (decentralized control). Methodologies using this decomposition are the hierarchical control (SIN-78a, SIN-78b), the decentralized control (SIN-81, TIT-83b and 86) and the steady disturbances method.

I.3.2. Vertical or Temporal Decomposition

The control task here is vertically divided into elementary control tasks, into "control levels". The following levels are commonly distinguished (figure 1.1):

-regulation or direct control,
-optimization (determination of regulator set points),
-adaptation (self-adaptation of model or of control law directly),
-self-organization (selection of model structures), of the control as a function of the environment).

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The control levels function with different time scales; if T_i is the intervention period of level i, then we have

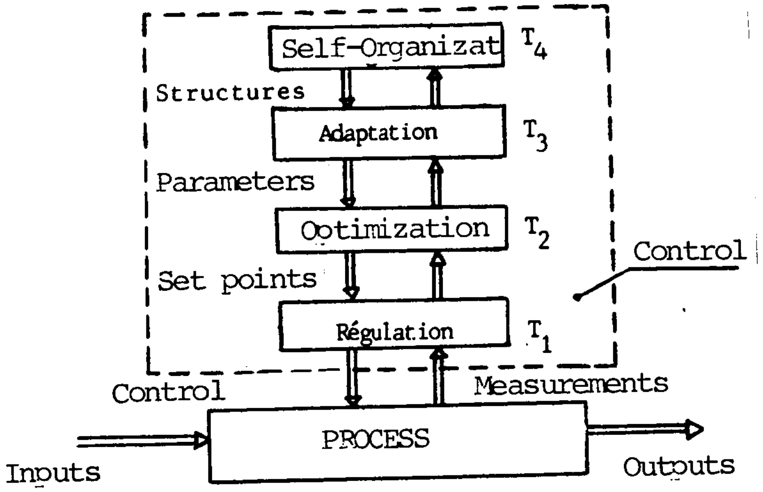

Fig. 1.1 - Control Functions Hierarchy

$T_1 < T_2 < T_3 < T_4$. The vertical decomposition essentially includes the complexity of the control function to be incorporated. It is used as a hierarchical control (TIT-79) and is present in the theory of singular disturbances (FOS-81, FOS-84, KOK-76)

I.3.3. Hybrid Decomposition (Spatio-Temporal)

This decomposition is a combination of the two preceding decompositions, and is interesting for geographically distributed systems which have several dynamics.

I.4. METHODS OF DIMENSION REDUCTION

These methods transform the model of the large scale system (n) into a small scale model ($n_1 \ll n$) upon which conventional methods are applicable. Among the most important methods in this group let us mention Aggregation which was initially developed for static models in economics (CHI-76) (AOK-68) and then generalized to dynamics models (BEM-79). It makes it possible to rediscover several reduction techniques already proposed in literature individually. The reduced models obtained by these methods may be used for analysis (stability, etc.), simulation, monitoring and calculation of suboptimal controls. Some of the structural properties of the original system may be incorporated in these models.

I.5 MULTI-CRITERIA OPTIMIZATION

In certain situations, the conventional single-criterion optimization is not valid, such as in the following cases:

A single control unit tries to satisfy conflicting objectives: minimization of production cost, maximization of production, or maximization of performance and minimization of control structure costs.

Several control units affect a single system. This is the case of an electrical power network made of of several subnetworks, each under the responsibility of a different administrative authority.

These problems may be formulated in terms of multi-criteria optimization involving different concepts of equilibrium (BAP-80).

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Before bringing this chapter to a close, let us return to two important cases: hierarchical control and decentralized control.

I.6 HIERARCHICAL CONTROL

A hierarchical control system is essentially made up of control units arranged in a hierarchy, in a pyramidal structure, affecting the control process, itself made up of interconnected subsystems (SIN-78a, SIN-78b).

The control units are divided into two or more levels, and some have only indirect access to the process to be controlled.

These units receive and process information from higher units. They then control other units which are lower than them in the hierarchy and send information to the higher levels to execute their tasks.

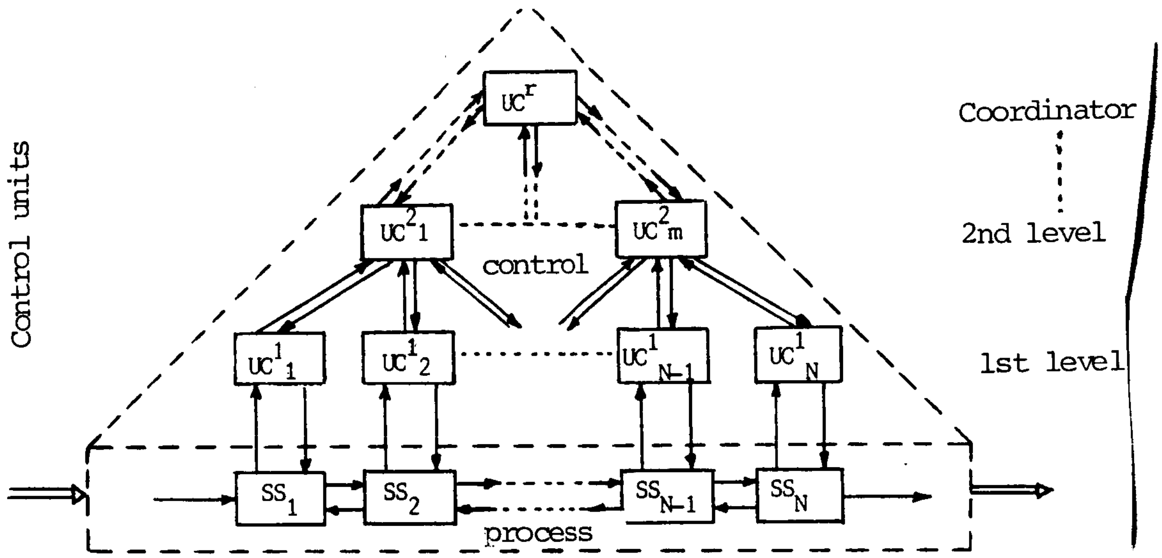

Fig. 1.2. - Multi-Level Structure

As we go up this hierarchy, the vision of reality expands, but the imprecision of the models increases (fig. 1.2).

I.7 DECENTRALIZED CONTROL

A control structure is called "Decentralized" if and only if all local controls are calculated only as an explicit function of local information (states, outputs, etc.) (Fig. 1.3) (SIN-83), (TIT-86).

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This is a structure with one control level, no control unit has an overall view of the entire process. Each unit issues a local control, assigning each subsystem with a functioning which

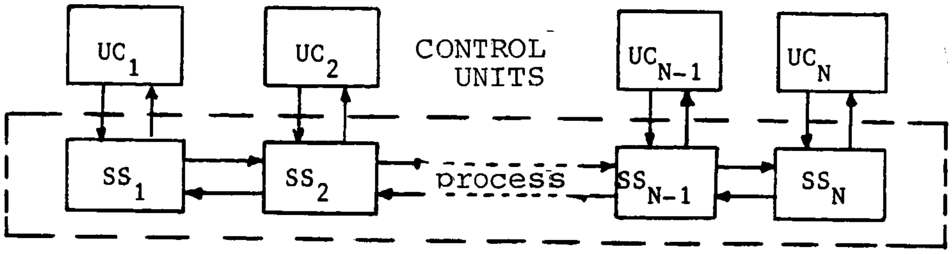

Fig. 1.3. - Decentralized Control Structure

will provide for satisfactory behavior of the overall system (stability, subcommittee with respect to a certain criterion, etc.).

Let us recall here a few of the reasons for using a decentralized control for complex systems:

R1 The construction of a central control unit, and transmission lines between the various subprocesses is costly, particularly for geographically divided systems (electrical power network, for example).

R2 The transmission of a signal between the various control units can only delay and distort the transfer of information.

R3 The instrumentation necessary for carrying out a decentralized control is simple.

R4 As a result of the technological growth of micro-computing, the cost of local control units is constantly decreasing.

R5 Decentralized control enables the local control units to be completely independent, and thereby increases the robustness of the system.

These reasons give an idea of the advantages offered by decentralized control in controlling complex processes. However,, this control creates new theoretical problems, in particular those associated with the FIXED MODES introduced by decentralization (WAN-73b) (SIN-81) (TIT-83B and 86) (TRA-84b). The latter point is the subject of our thesis, and it will be studied in detail in the following chapters according to the following plan:

Let us begin the second chapter by recalling the fundamental notions of controllability and observability, and the conditions for the existence of a solution to the problem of stabilization and the pole placement by centralized control. Let us then formulate the problem of stabilization and of free pole placement
by a decentralized control, and let us give the main results to 7
this problem by introducing the notion of FIXED MODES. This concept is fundamental. In effect, free pole placement, possible only in the absence of fixed modes and in the presence of unstable fixed modes, makes stabilization impossible.

After stressing the importance of the notion of fixed modes, chapters III, IV and V discuss the methods of characterizing these modes. The 3rd chapter summarizes the different methods of characterizing fixed modes algebraically in the time and frequency systems and shows the links between the different characterizations. Furthermore, we propose for the time system, a characterization of the fixed modes based on the notion of the sensitivity of the natural values, and in the frequency system a necessary condition for the existence of fixed modes. We also give a physical interpretation of fixed modes.

In chapter 4, after reviewing the notions of structural controllability and observability, two types of fixed modes are brought to light: STRUCTURAL FIXED MODES (structurally uncontrollable modes in a decentralized manner) and NONSTRUCTURAL FIXED MODES, then we introduce the algebraic characterizations of structural fixed modes and apply our approach by sensitivity to the characterization of such modes.

The 5th chapter discusses the graphic characterization methods of the two types of fixed modes.

Chapter VI is devoted to the problems of decentralized stabilization in the presence of fixed modes. It is shown that an unsteady decentralized control may stabilize the system if the fixed modes are nonstructural. We then give a summary of decentralized stabilization methods, using feedback laws varying in time, and propose the use of a "Vibrational Control" to stabilize the unstable fixed modes of the system. We therefore establish the existence of vibrational and decentralized feedback laws permitting the stabilization of the nonstructural fixed modes of the system. The stabilization of structural fixed modes by the vibrational system is also discussed.

The 7th chapter introduces a summary of methods of selecting a control structure (static or dynamic) permitting a free pole placement of a system. Furthermore, four procedures are developed to find an optimal control structure avoiding fixed modes.

The 8th chapter is concerned with algorithms for summarizing decentralized controls in the absence of fixed modes. After presenting a summary of most algorithms which exist in literature, we add parametric robustness considerations, formulate the problem of the corresponding optimization and propose an algorithm for calculating a parametrically robots control (minimization of the sensitivity of the quadratic criterion with respect to the variations of the system parameters), using the projected gradient method.

Finally, the last chapter introduces the results of applying the methodologies proposed in the various chapters to a model for a ship steam generator.

I.8 CONCLUSION

In this introductory chapter, we gave a brief summary of the methods of controlling complex systems, specifying the strict framework in which we are in, i.e. "THE DECENTRALIZED CONTROL OF LARGE SYSTEMS: FIXED MODES, SENSITIVITY AND PARAMETRIC ROBUSTNESS", after giving a few reasons for using such a control.

CHAPTER II - POLE STABILIZATION AND PLACEMENT IN LINEAR SYSTEMS

II.1 INTRODUCTION

After briefly recalling the fundamental notions of controllability and observability of steady linear systems, this chapter discusses the problems of pole stabilization and placement of linear systems under structural constraints, by introducing the notion of fixed modes, and providing the necessary and sufficient conditions for the existence of a solution to these problems (in the time and frequency systems).

II.2 CONTROLLABILITY AND OBSERVABILITY

Let us consider a system which is dynamic, linear, multivariable, continuous and invariant in time, described by: /10

$$\overset{\circ}{X}(t) = A\, X(t) + B\, U(t) \qquad (2\text{-}2\text{-}1a)$$

$$Y(t) = C\, X(t) \qquad (2\text{-}2\text{-}1b)$$

where $X \in R^n$, $U \in R^m$ and $Y \in R^p$ are state, control and output vectors respectively, A, B and C are constant matrices of the appropriate dimensions.

II.2.1. Notion and Criterion of Controllability

A state (x_0, t_0) is said to be "CONTROLLABLE" if a finite time t_i can be found that is greater than t_0, and an input U(t) in the interval [t_0,t_1] which converts the state into (x_1, t_1). If all states of a system are controllable, the system (or the pair (A,B) is said to be totally controllable. This leads to a decomposition of the state vector into two subvectors equations (2-2-1) which are expressed in the following form:

$$\begin{bmatrix} \overset{\circ}{X}_G \\ \overset{\circ}{X}_{ING} \end{bmatrix} = \begin{bmatrix} A^*_{11} & A^*_{12} \\ 0 & A^*_{22} \end{bmatrix} \begin{bmatrix} X_G \\ X_{ING} \end{bmatrix} + \begin{bmatrix} B^*_1 \\ 0 \end{bmatrix} U$$

For all systems under consideration, the following theorem gives a necessary and adequate condition of controllability:

Theorem 2-2-1 (FOS-72) (KAI-62)

The system (2-2-1) is controllable if and only if it is in the equivalent form:

1) Kalman Criterion (KAL-62)

$$\text{Rank } Q_g = \text{Rank } [B,\ AB,\ \ldots,\ A^{n-1} B] = n \qquad (2\text{-}2\text{-}2)$$

2) Popov - Belevit - Hautus (PBH) Criterion

The scalar products $\langle W_i, b_j\rangle$ are not zero for any j, as b_j is the jth column of B and W_i are the natural vectors to the left of A*.

The Kalman criterion is not minimal and most often, it may be verified whether the matrix:

$$Q_g = [B,\ AB, \ldots,\ A^{\nu_c - 1} B]$$

*The natural values to the left W and to the right, related to the natural value s are given by:

$$(sI-A)\ V = 0$$
$$W^T\ (sI-A) = 0$$

is of the same rank as n whereas ν_c is less than n. ν_c is called CONTROLLABLE INDEX". The PBH criterion is in practice simpler to use, since it may be expressed in the following form: the system (2-1) is controllable if and only if:

$$\text{Rank } (sI-A,\ B) = n \text{ for any } s \in \sigma\ (A) \qquad (2\text{-}2\text{-}3)$$

II.2.2. Notion Criterion of Observability

A state (X_O, t_O) is said to be "OBSERVABLE" if the initial state x_O may be identified based on knowledge of the output Y(t) and input U(t) on an interval [t_O,t1]. As with controllability, the system is said to be completely observable if all of its states are observable. In the sense of observability, the state vector may be divided into two subvectors, one being observable: X_O and the other unobservable: X_{INO}. It is shown that the state equations (2-2-1) may be expressed in the form:

$$\left.\begin{aligned} \begin{bmatrix} \dot{X}_O \\ \dot{X}_{INO} \end{bmatrix} &= \begin{bmatrix} A'_{11} & 0 \\ A'_{21} & A'_{22} \end{bmatrix} \begin{bmatrix} X_O \\ X_{INO} \end{bmatrix} + \begin{bmatrix} B'_1 \\ B'_2 \end{bmatrix} u \\ \begin{bmatrix} Y \end{bmatrix} &= \begin{bmatrix} C'_1 & 0 \end{bmatrix} \begin{bmatrix} X_O \\ X_{INO} \end{bmatrix} \end{aligned}\right| \qquad (2\text{-}2\text{-}4)$$

In a dual manner, the necessary and adequate condition of observability is given by the following theorem:

Theorem 2-2-2 (FOS-72) (KAI-80)

The system (2-2-1) is observable if and only if, in equivalent form:

1) Kalman Criterion (KAL-62):

$$\text{rank } Q_O = \text{rank}\left[C^T, A^TC^T, \ldots, (A^T)^{n-1} C^T\right] = n \qquad (2\text{-}2\text{-}5)$$

2) Popov - Beleviat - Hautus (PHB) Criterion:

The scalar products $<c_j, v_i>$ are not zero for any j, c_j being the C lines and v_i the natural vectors to the right of A.

As is the case for the controllability, if

$$\text{rank } Q_O = \text{rank}\left[C^T, A^TC^T, \ldots, (A^T)^{\nu_o - 1} C^T\right] = n$$

is called "OBSERVABILITY INDEX". The PBH criterion may also take the following simple form: the system (2-2-1) is observable if and only if:

$$\text{Rank } \begin{bmatrix} sI-A \\ C \end{bmatrix} = n \text{ for any } s \in \sigma\ (A) \qquad (2\text{-}2\text{-}6)$$

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II.2.3. Canonical Structure of a System (KAL-62)

Based on the above and from an overall standpoint the state vector may be divided into four components X_1 to X_4 so that:

$X_1 \in R^{n_1} = X_G \cap X_O$	(controllable and unobservable states)
$X_2 \in R^{n_2} = X_G \cap X_{INO}$	(controllable and observable states)
$X_3 \in R^{n_3} = X_{ING} \cap X_O$	(incontrollable and unobservable states)
$X_4 \in R^{n_4} = X_{ING} \cap X_{INO}$	(incontrollable and observable states)

with $n = n_1 + n_2 + n_3 + n_4$, Kalman (KAL-62) showed that a real, steady conversion matrix exists making it possible to switch from model (2-2-1) to the model below.

The structure of the system thus demonstrated by (2-2-7) may be illustrated by figure (2.1). Note that the block modes A_{22} are completely controllable and observable, and therefore verify

$$\begin{bmatrix} \overset{\circ}{X}_1 \\ \overset{\circ}{X}_2 \\ \overset{\circ}{X}_3 \\ \overset{\circ}{X}_4 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} U \qquad (2\text{-}2\text{-}7a)$$

$$\begin{bmatrix} Y \end{bmatrix} = \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix} \begin{bmatrix} X_1^T & X_2^T & X_3^T & X_4^T \end{bmatrix}^T \qquad (2\text{-}2\text{-}7b)$$

according to the PBH criterion:

Rank

$$\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix} = n$$

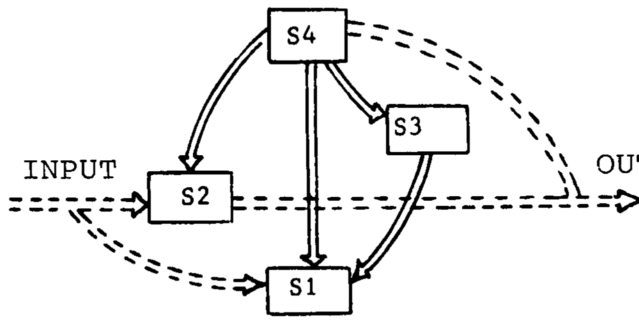


Fig 2.1 - Canonical Structure of a Linear System

II.22.4. Origin of Uncontrollability and Unobservability

On the basis of an example consisting of several single-entry single-output subsystems, (FOS-78f) shows that the appearance of incontrollable and/or unobservable modes is the result of a compensation (simplification) between a subsystem and a zero of another, according to the correspondence:

Zero upstream = upstream pole $\longrightarrow$ uncontrollability
Upstream pole - downstream zero $\longrightarrow$ unobservability

II.2.5. Practical Importance of the Notions of Controllability and Observability (FOS-78)

1) An uncontrollable mode is not connected to the input and the feedback of this mode will change only as a function of its dynamics and the initial condition assigned to it, independently of the control.

2) If a mode is unstable, uncontrollable, but observable, the output is unstable. The fact that it is uncontrollable therefore excludes any possibility of stabilizing the system. In this case the problem is therefore not to search for a control law, but to modify the structure of the system.

3) Let us assume that the system has a controllable, but unobservable mode, and this mode is not connected to the output. A stable output will be observed, but the instability internal to the system will risk leading to the rupture of the system or to a nonlinear operation (saturation) and in this case the nonlinear model adopted is no longer valid.

4) On the practical level, a distinction must be made between stable and unstable uncontrollable (unobservable) modes. The exact compensation of a pole by a zero (or conversely) is totally theoretical. When it is said that there is a pole/zero compensation, actually a dipole is created. If this dipole is in the left half plane (near the value $-\alpha$), it will contribute to a term of the form $\varepsilon e^{-\alpha t}$, ε being the residue associated with the pole (very low), and this term amy be disregarded. If the dipole is in the right half plane, we will have an unstable plane as small as ε.

The results 1, 2 and 3 may be extrapolated with an output of the system (2-2-7), i.e. U = KY, the typical equation of the system in a closed loop becoming:

$$\det(sI_{n_1}-A_{11})\cdot\det(sI_{n_2}-A_{22}-B_2K_2C_2)\cdot\det(sI_{n_3}-A_{33})\cdot\det(sI_{n_4}-A_{44}) = 0$$

This equation shows quite well that the uncontrollable and/or unobservable modes cannot be displaced regardless of the value of the K control applied. These modes shall be called "CENTRALIZED FIXED MODES".

II.3 STABILIZATION AND PLACEMENT OF CENTRALIZED CONTROL POLES

The problem is presented as follows: given a supposedly unstable system (2-2-1), find a control that stabilizes the system, or that places the system modes in a specified region of the complex plan. Wonham (WON-67) showed that if all states of the system are measurable, then the system can be stabilized with a static state feedback if and only if the uncontrollable modes of the system (centralized fixed modes) are stable, and a system pole placement may be achieved if it is totally controllable (no centralized fixed modes).

In practice, unmeasurable states often exist (in particular /14
for a large system that has a large number of states). In this case the system has to be controlled by an output feedback. Brasch and Pearson (BRA-70) showed that the system can be stabilized by a dynamic output feedback if and only if its uncontrollable and/or unobservable modes are stable, and the pole placement can be achieved if and only if the system is totally controllable and observable (absence of centralized fixed modes). The minimum order of the dynamic compensator required is: min (ν_g-1, ν_o-1), where ν_g and ν_o are the controllability and observability indices respectively. It should be pointed out here that for a general case the problem of stabilization or pole placement by static output feedback is not totally resolved.

II.4 STABILIZATION AND POLE PLACEMENT OF SYSTEMS UNDER STRUCTURAL STRESS

This involves controlling a system by introducing stresses on the structure of the control system. Two stress cases are to be considered: 1 a local control station only has local information (states, outputs and possibly local external inputs). In this case the control is "DECENTRALIZED" (figure 2.2.a). 2 The structural stresses are of any type, namely one local station may have "other information" from other a subsystems in addition to available local information, but all stations do not simultaneously have information on all subsystems (as in the case of a centralized control) (figure 2.2b).

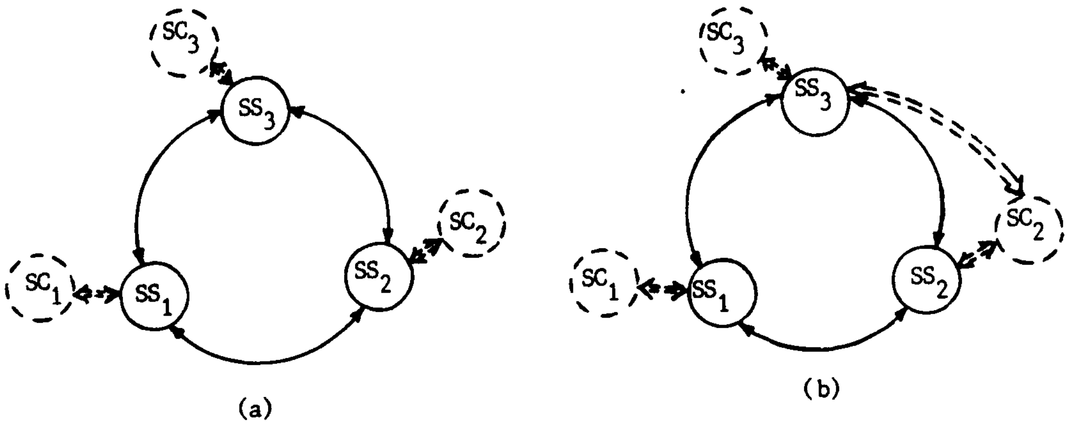


Fig. 2.2 - Example of a system (N=3) under structural stress
a) decentralized - b) any type

II.4.1 - Problem of a Decentralized Control by Output Feedback

Let us consider a dynamic, continuous, linear, multivariable system with N local control stations, described by:

$$\overset{\circ}{X}(t) = A\, X((t) + \sum_{i=1}^{N} B_i\, u_i(t) \qquad (2\text{-}4\text{-}1a)$$

$$y_i(t) = C_i\, X(t) \qquad i=1,\ldots,N \qquad (2\text{-}4\text{-}1b)$$

$X(t) \in R^n$ is the state, $u_i(t) \in R^{m_i}$ and $y_i(t) \in R^{p_i}$ are the controls and local outputs of the i-th subsystem. A, B_i and C_i are the constant matrices of the appropriate dimensions.

Overall, the system is expressed:

$$\overset{\circ}{X}(t) = A\ X(t) + B\ U(t) \qquad (2\text{-}2\text{-}2a)$$

$$Y(t) = C\ X(t) \qquad (2\text{-}2\text{-}2b)$$

with $U(t) \in R^m$; $Y(t) \in R^p$, where $\sum_{i=1}^{N} m_i$ and $p = \sum_{i=1}^{N} p_i$, and

$$B = [B_1, \dots, B_N]$$

$$C^T = [C_1^T, \dots, C_N^T]$$

In the frequency range, the input-output ratio using the system transfer matrix is described by:

$$\begin{bmatrix} y_1(s) \\ \vdots \\ y_N(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & \cdots & G_{1N}(s) \\ \vdots & & \vdots \\ G_{N1}(s) & \cdots & G_{NN}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ \\ u_N(s) \end{bmatrix} \qquad (2\text{-}4\text{-}3)$$

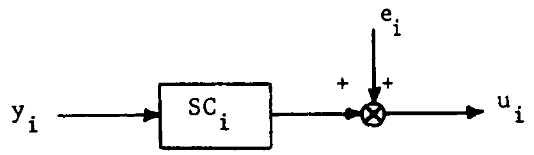
i.e. globally:

$$Y(s) = G(s)\ U(s) \qquad (2\text{-}2\text{-}4)$$

The problem consists of finding a decentralized control (N local controls) using output feedback in order to stabilize the overall system. These controls are generated by the following local dynamic compensators:

$$\begin{aligned} \overset{\circ}{z}(t) &= S_i\ z_i + R_i\ y_i(t) \\ u_i(t) &= Q_i\ z_i + K_i\ y_i(t) + e_i(t) \qquad i=1,\dots,N \end{aligned} \qquad (2\text{-}4\text{-}5)$$

with $z_i(t) \in R^{\xi_i}$ & $e_i(t) \in R^{m_i}$ state and external input of the i-th local control system. S_i, R_i, Q_i and K_i are the real, constant matrices with the appropriate dimensions.



The control system is globally expressed as follows:

$$\overset{\circ}{Z}(t) = S\,Z(t) + R\,Y(t) \qquad (2\text{-}4\text{-}6)$$

$$U(t) = Q\,Z(t) + K\,Y(t) + E(t)$$

with S, R, Q and K block-diagonal matrices with the appropriate dimensions:

S - block diag. (S_i)
R = block diag. (R_i)
Q - block diag. (Q_i)
K - block diag. (K_i)

and $E(t) \in R^m$ the external control vector with $E^T = [C_1^T, \ldots, C_1^T]$.

When the control (2-4-6) is applied to the system (2-4-2), the closed loop system is expressed by:

$$\left[\begin{array}{c} \overset{\circ}{X} \\ \hline \overset{\circ}{Z} \end{array}\right] = \left[\begin{array}{c|c} A+BKC & BQ \\ \hline RC & S \end{array}\right] \left[\begin{array}{c} X \\ \hline Z \end{array}\right] + \left[\begin{array}{c} B \\ \hline O \end{array}\right] E \qquad (2\text{-}4\text{-}7)$$

The set of local controls should be selected so that the system (2-4-7) is asympotically stable, i.e. all of these poles are in the left half of the complex plan.

II.4.2. Notion of Decentralized Fixed Modes

Owing to the structural stresses, satisfaction of centralized stabilization condition of the system does not lead to a decentralized stabilization of it. Based on an example, Wang (WAN-78b) showed for the case of a state feedback that even if the system (A,B) and each subsystem (A_{ii}, B_{ii}) are completely controllable and observable (absence of centralized fixed modes) it may be impossible to stabilize the system using a decentralized control. This impossibility is due to the existence of DECENTRALIZED FIXED MODES. Such a notion was introduced by Wang and Davison (WAN-73b).

Definition 2-4-1 (WAN-73b) Decentralized Fixed Polynomial

Let us consider the system (2-4-2) and the set of diagonal control block matrices defined as follows:

$$K_d = \{K/K = \text{block.diag}(K_i),\ K_i \in R^{m_i x p_i},\quad i=1,\ldots,N\} \qquad (2\text{-}4\text{-}8)$$ /17

The highest common divider of typical polynomials for (A+BKC), for $K \in K_d$ is defined as a decentralized fixed polynomial of the system with respect to the set of control matrices K_d, and is notated:

$$\Psi\ (s;\ C,\ A,\ B,\ Kd) = \underset{K \in K_d}{\text{p.g.c.d.}} \{\det(sI-A-BKC)\} \qquad (2\text{-}4\text{-}9)$$

Definition 2-4-2 (WAN-73b) Decentralized Fixed Modes

For the system (C,A,B) and the set K_d of block diagonal matrices, the set of fixed system modes with respect to Kd is defined by:

$$\Lambda\ (C,A,B,K_d) = \bigcap_{K \in K_d} \sigma\ (A+BKC) \qquad (2\text{-}4\text{-}10a)$$

where $\sigma(M)$: set of natural values of M. Similarly, the decentralized fixed modes may be defined as the roots of the decentralized fixed polynomials, namely:

$\Lambda\ (C,A,B,Kd) = \{s/s \in C \text{ and } \psi(si,C,A,B,Kd) = 0\}$ (2-4-10b)

Since the set Kd contains all block-diagonal matrices, it therefore contains the zero matrix and therefore:

$$\Lambda\ (C,A,B,K_d) \subset \sigma(A) \qquad (2\text{-}4\text{-}11)$$

The relationships (2-4-10) and (2-4-11) may be illustrated by figure 2.3.

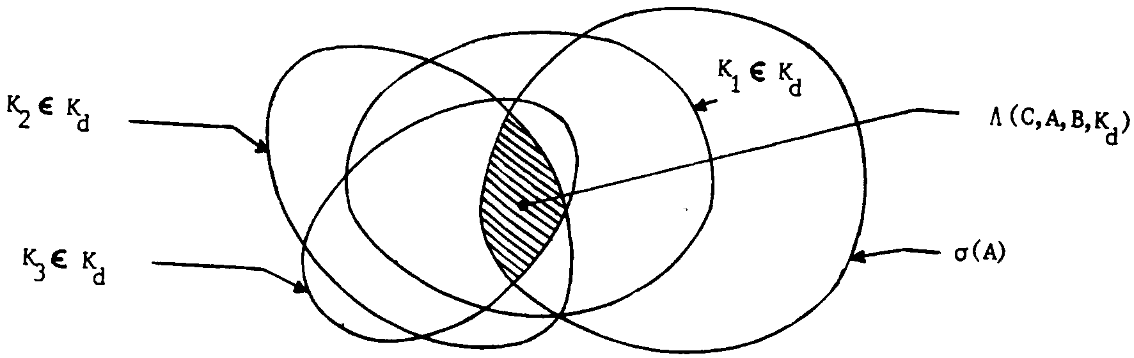


Fig. 2.3

Let us notate the centralized fixed modes by $\Lambda_c(C,A,B,R^{mxp})$ and the fixed modes due to decentralization by Λ_d; the fixed modes of a system with respect to a matrix $K \in Kd$ are therefore $\Lambda = \Lambda_c + \Lambda_d$. Consequently, we will consider that the system (C,A,B) is completely controllable and therefore observable, $\Lambda_c = \{\emptyset\}$, and the fixed modes will not be only due to decentralization, i.e. the fixed modes will be only related to the control structure.

II.4.3 Stabilization and Pole Placement OF DECENTRALIZED SYSTEMS /18

Many researchers have studied the problem of stabilization and pole placement of decentralized systems in a deterministic formulation (AOK-72, WAN-73b, COR-76a and 76b, KOB-78 and 82, FES-79 and 80, POT-79, KUL-82, etc...), but the distinguishing results of all of these studies showing the significance of decentralized fixed modes are essentially due to Wang and Davison (WAN-73b) and Corfmat and Mores (COR-76a and 76b).

II.4.3a. Results by Wang and Davison (WAN-73b)

The results in a closed loop are expressed by the following theorem:

Theorem 2-4-1 (WAN-73b)

i) the closed loop system as defined in (2-4-7) is stabilizable by appropriately selecting the matrices Ri, Si, Qi and Ki, i=1,...,N if and only if $\Lambda\,(C,A,B,K_d) \subset \mathbf{C}^-$, where C- is the left half of the complex plan and K_d the set defined in (2-4-8).

ii) All of the poles of the closed loop system defined in (2-4-7) may be placed in C_g if and only if $\Lambda\,(C,A,B,K_d) \subset \mathbf{C}_g$. where C_g is a symmetrical area of the complex plan.

Corollary 2-4-1

An arbitrary pole placement of the system may be achieved if and only if $\Lambda(C,A,B,K_d) = \{\emptyset\}$.

<u>Remark</u>: The notion of decentralized fixed modes is therefore a generalization of uncontrollable and/or unobservable fixed modes in a centralized control

Example 2-4-1

Given the system below:

Example 2-4-1

$$\overset{\circ}{X} = \begin{bmatrix} -2 & & \\ & a & \\ & & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u_2$$

$$y_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} X$$

$$y_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} X$$

Is it possible to control this system with a completely decentralized output feedback? Given U = KY = diag $(k1,k2).Y$; the typical polynomial of the closed loop system is:

$$\det(sI-A-BKC) = (s+2-k_1)\ (s-a)\ (s+1-k_2)$$

/19

and according to the definitions (2-4-1) and (2-4-2) the system has a decentralized fixed mode at $s = a$ and the fixed polynomial is $(s-a)$. According to the theorem (2-4-1), if $a < 0$, the system is stabilizable by decentralized feedback, but if $a > 0$ the system is not.

If the matrix A of the system becomes:

$$A = \begin{bmatrix} -2 & & \\ & a & \\ & 1 & -1 \end{bmatrix}$$

the typical closed loop polynomial is expressed:

$$\det(sI-A-BKC) = (s+2-k_1)\ [(s-a)\ (s+1-k_2)\ -\ k_2]$$

and the system does not have fixed modes, its poles may therefore be arbitrarily placed in the complex plan.

II.4.3b. Results by Corfmat and Mores (COR-76)

Corfmat and Mores (COR-76a and 76b) used a geometrical approach (WON-74) to handle the problem and obtained more constructive results based on the question: Do local static controls exist in the form:

$$u_i = K_i\, y_i + e_i \qquad i=1,\ldots,N$$

which, when applied to the system (2-4-1), make the closed loop system:

$$\overset{\circ}{X}(t) = (A + \sum_{j=1}^{N} B_j\, K_j\, C_j)\, X(t) + \sum_{i=1}^{N} B_i\, e_i$$

$$y_i(t) = C_i\, X(t) \qquad i=1,\ldots,N$$

controllable and observable from a single station (i.e. j) from which the centralized control algorithms (stabilization or pole placement) may be used to calculate a dynamic control of type (2-4-6)? Figure 2.4 gives an illustration of this step on a system with three control stations, controllable and observable by the third of these.

To have a clearer understanding of these results, a few definitions are necessary:

Definition 2-4-3 (COR-76b) Complementary Subsystem

(C_β, A, B_α) is a complementary subsystem of the system (C,A,B) defined in (2-4-2) if α and β are two natural subsystems, not void of the set $\mathbf{N}$ where $\mathbf{N} = \{1,\ldots,N\} = \alpha \cup \beta$

with
$$\mathbf{N} = \{1,\ldots,N\}$$
$$\alpha = \{1,\ldots,i_k\}$$
$$\beta = \{i_{k+1},\ldots,k_N\}$$

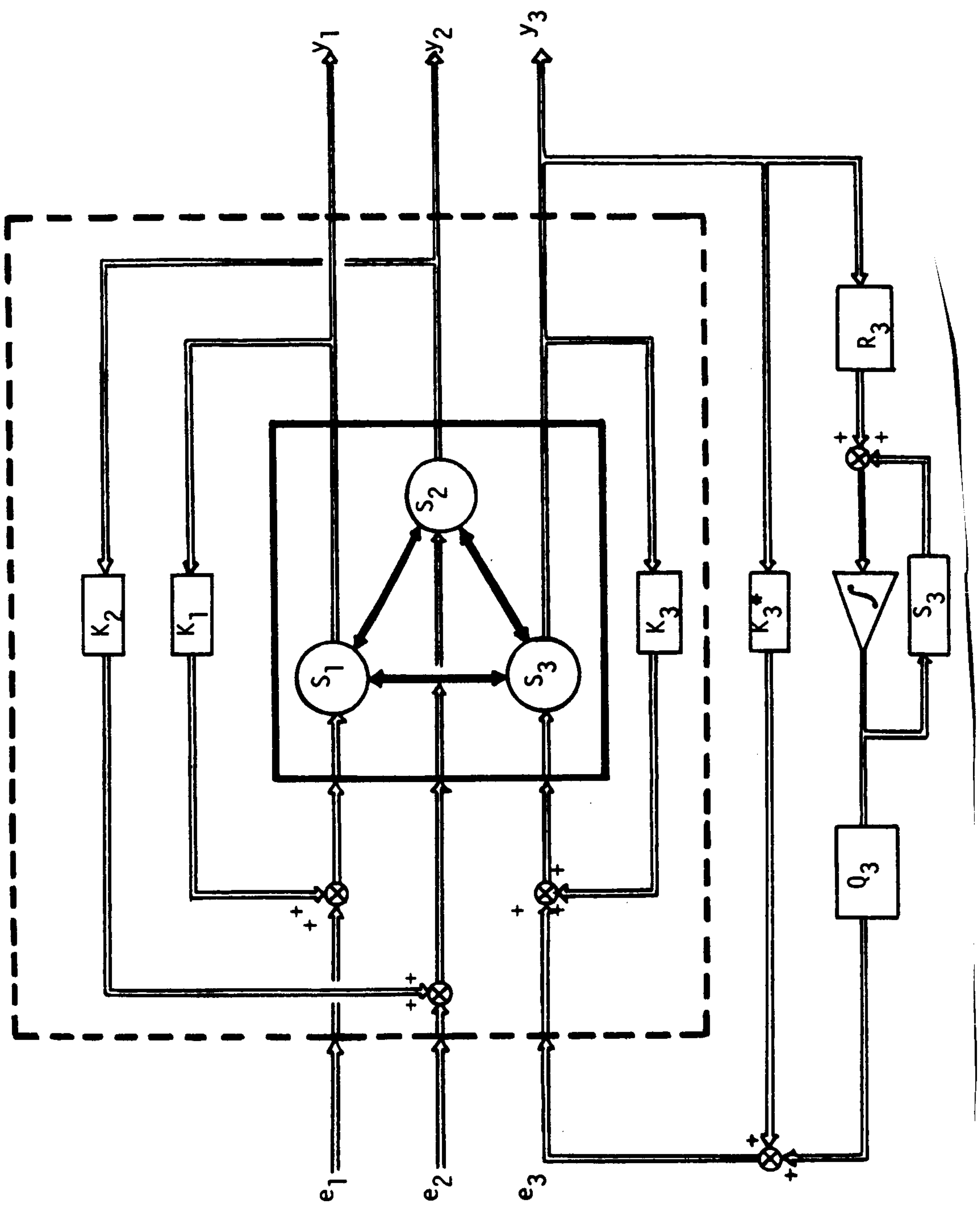


Fig. 2.4

and where

$$B_\alpha = \left[B_{i_1}, \ldots, B_{i_k}\right]$$
$$C_\beta = \left[C^T_{i_{k+1}}, \ldots, C^T_{i_N}\right]^T$$

For a system with N control stations, there are $2^N - 2$ complementary subsystems. For example, if N = 3, we have the following 6 complementary subsystems: (C_2C_3,A,B_1), (C_1C_3,A,B_2), (C_1C_2,A,B_3), (C_3,A,B_1B_2), (C_2,A,B_1B_3) & (C_1,A,B_2B_3).

Definition 2-4-4 (COR-76b) Highly Connected System

A (C,A,B) system with N subsystems is said to be highly connected if:

$$G_{\beta\alpha} = C_\beta\,(sI-A)^{-1}\,B_\alpha \neq 0 \qquad \forall\alpha$$

where $G_{\beta\alpha}$ represents the transfer matrix of a complementary subsystem (C_β,A,B_α); $G_{\beta\alpha} \neq 0$ implies that the output of the aggregated station β is influenced by the control of the aggregated station α.

Definition 2-4-5 (COR-76b) "Remnant Polynomial"

The "remnant polynomial" of a system (C,A,B) is described by

$$R(C,A,B) = \begin{cases} 1 & \text{if} \quad r \geq t \\ \prod_{i=r+1}^{t} T_i(s) & \text{if} \quad r < t \end{cases}$$

where r is the rank of the rational transfer matrix $C(sI-A)^{-1}B$, and $T_1,\ldots,T_t(s)$ are transmission polynomials of (C,A,B) (MOR-73) (Appendix 1). In other words (ROS-70) the "remnant polynomial" is the product of n (system order) first invariant polynomials of the matrix $\begin{bmatrix} s-IA & B \\ C & 0 \end{bmatrix}$

Definition 2-4-6 (COR-76) Complete System

i) A system (C,A,B), with a single control (N=1), is complete if its transfer function is not zero and its "remnant polynomial" is equal to 1, i.e.: $C\ (sI-A)^{-1}\ B \neq 0$ and $R\ (C,A,B) = 1$.

ii) A system (C,A,B), with N control stations, is complete if all of this complementary subsystem are complete, i.e. the system is highly connected and $R(C_\beta, A, B_\alpha) = 1 \quad \forall\ \alpha$

Theorem 2-4-2 (COR-76b)

If the system (C,A,B) defined in (2-4-2) is controllable, observable and highly connected, then with a decentralized system:

i) the system may be stabilized if and only if the "remnant polynomials" of all complementary subsystems are stable.

ii) all poles of the system may be assigned if and only if the system is full.

/22

Theorem 2-4-3 (COR-76b)

If the system (C,A,B) defined in (2-4-2) is controllable and observable, then with a decentralized control:

i) the system may be stabilized if and only if the poles of the system not belonging to the pole union of highly connected systems are stable and stabilization is possible for each highly connected subsystem.

ii) all poles of the system may be placed if and only if the sum of the dimensions of highly connected subsystems is equal to the dimension of the system and the poles may be placed for each highly connected subsystem.

Corfmat and Morse (COR-76) gave an explicit solution to the problem of stabilization and decentralized pole placement. Their analysis is attractive, but from a practical standpoint, their approach has a few drawbacks:

a) even if all modes of a complex system are controllable and observable by a single station, there may be a few modes that are not very controllable and observable by this station, making it necessary for large, but not very practical gains.

b) it is not certain that all available degrees of freedom are used to design the control.

c) "reducing" the complexity of the overall system into a single station is generally not desirable.

Despite these drawbacks, other researchers have become interested in the problem as formulate by Corfmat and Morse (COR-76). For example Kobayashi et al (KOB-78), Potter et al (POT-79), and Fessas (FES-79,80,82a and 82b). Kobayashi et al (KOB-78) used a local control varying in time, Potter et al (POT-79) presented for a system with 2 control stations, the findings of Corfmat and Mores (COR-76) in the form of a rating test of the system matrix. Such a result is very important, because it is directly related (in its general form) to the fixed modes characterization (AND-81a). Finally, Fessas (FES-79, 80, 82a and 82b) use polynomial matrices ("matrice fraction description" (ROS-70) (WOL-74) and comes to algebraically equivalent results (FES-81) to those of Corfmat and Morse (COR-76b).

Let us note that the findings of Corfmat and Morse (COR-76b) reveal the notion of transmission zero (Appendix 1) through the

"remnant polynomial". If a subsystem has a transmission zero, the "remnant polynomial" will have one also. The system will therefore not be stabilizable by a decentralized control. In /23
In the next chapter we will show how the findings by Wang and Davison (WAN-73b) coincide with those of Corfmat and Morse (COR-76).

Example 2-4-2 (KAT-81)

Given a globally controllable and observable system, described by:

$$\dot{X} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

$$y_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

The transfer function of the system is:

$$G(s) = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] = \left[\begin{array}{c|c} (s-1)^{-1} & -s^{-2}(s-2)^{-1} \\ 0 & s^{-1}(s-2)^{-1} \\ (s-1)^{-1} & 0 \\ \hline (s-1)^{-1} & 0 \end{array}\right]$$

The system has two complementary subsystems verifying:

$$(C_1, A, b_2) \longrightarrow G_{12}(s) = C_1(sI-A)^{-1} B_2 \neq 0$$

$$(C_2, A, b_1) \longrightarrow G_{21}(s) = C_2(sI-A)^{-1} B_1 \neq 0$$

The system is therefore highly connected.

The "remnant polynomials" of the complementary subsystems are:

$$R(C_1, A, B_2) = R(C_2, A, B_1) = 1$$

the system is therefore complete, and may be rendered controllable and observable by the 1st station by applying a static control to the second one. The system is therefore stabilizable by a decentralized control, and even its poles may be arbitrarily placed in the complex plan (complete system and "remnant polynomials" equal to 1).

II.4.4. Special Case of Interconnected Systems

The special feature of this system class is that subsystems are generally only interconnected by inputs and outputs (figure 2.5). /24

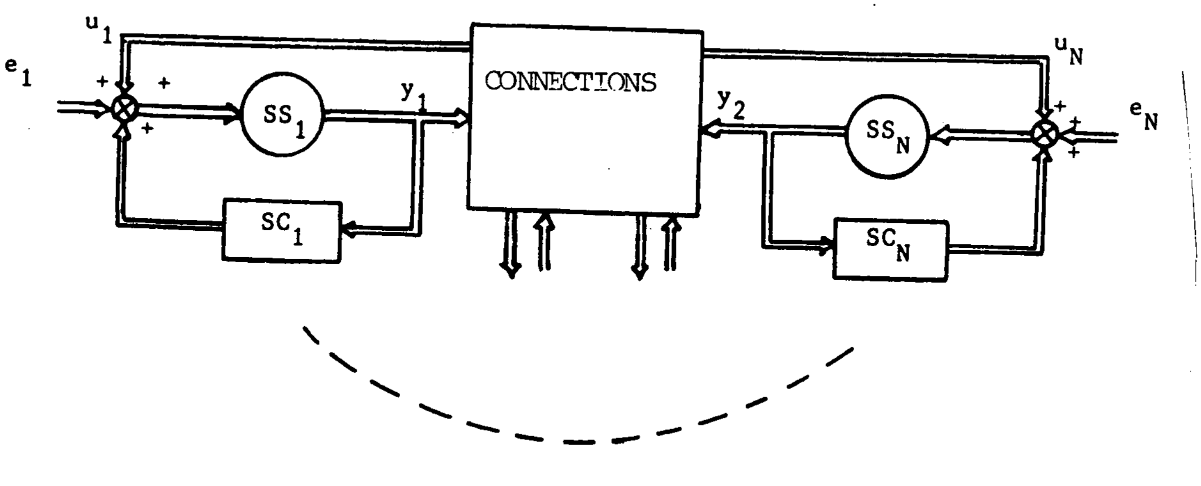


Fig. 2.5

The equation for the state of such a system is:

$$\overset{\circ}{X}_i(t) = A_i\ X_i(t) + B_i\ u_i(t) \qquad (2\text{-}4\text{-}12a)$$

$$y_i(t) = C_i\ X_i(t) \qquad i=1,\ldots,N \qquad (2\text{-}4\text{-}12b)$$

The interconnections are described by the following system:

$$\overset{\circ}{Z} = M\ Z + \sum_{j=1}^{N} L_j\ y_j \qquad (2\text{-}4\text{-}13a)$$

$$u_i = N_i Z + \sum_{j=1}^{N} K_{ij}\ y_j + e_i \qquad i=1,\ldots,N \qquad (2\text{-}4\text{-}13b)$$

where $X_i \in R^{n_i}$, $u_i \in R^{m_i}$, $y_i \in R^{p_i}$, $Z \in R^q$ and $e_i \in R^{m_i}$. A, B_i, C_i, Z, L_j, N_i, K_{ij}, are constant matrices of the appropriate dimensions. Let us note that if q =m 0, we obtain the following special case of static interconnections:

$$u_i(t) = \sum_{j=1}^{N} K_{ij}\ y_j(t) + e_i(t) \qquad (2\text{-}4\text{-}14)$$

The problem of stabilizing interconnected systems (2-2-12) by a dynamic system (2-4-13) (or by the special cases of (2-4-13), such as (2-4-14)), by local dynamic compensators of type (2-4-6), is studied by many researchers (SEZ-78; 80, SAE-79, DEC-81, DAV-79; 83 and 85b, RAM-82, GUA-83, LIN-84, AND-84, MIN-85, etc...). Linnemann (LIN-84) considered the general interconnection case (2-4-13) and gives the following adequate condition:

Theorem 2-4-4 (LIN-84) /25

An adequate condition for the existence of dynamic decentralized controls by type (2-4-6) output feedback which stabilize the (2-4-12) systems interconnected by a dynamic system (2-4-13) is:

1) the local system s(2-2-12) are stabilizable and

detectable, i.e.:

$$\text{Rank } (sI-A_i \; B_i) = n_i \text{ and Rank } \begin{bmatrix} sI-A_i \\ C_i \end{bmatrix} = n_i \qquad i=1,\dots,N$$

for all unstable modes A_i ($s \in C^+$)

2) the interconnection system (2-4-13) is stable.

Let us note that condition 1 may be replaced by a stronger condition of local controllability and observability. For the special case of type (2-2-14) static interconnections, the above theorem takes the following simple form:

Corollary 2-4-2

The systems (2-4-12) (2-2-14) are stabilizable by dynamic output feedbacks if the local system (2-4-12) are stabilizable by dynamic output feedbacks.

Saeks (SAE-79) comes to the same result showing that the set of decentralized fixed modes is identical to the set of centralized fixed modes, namely $\Lambda = \Lambda_d = \Lambda_c$. Consequently, the decentralized stabilization conditions (or pole placement) of the system are the same as for a centralized stabilization (absence of uncontrollable and/or unobservable modes) and the decentralized stabilization criterion may be reduced to the controllability and to the observability of local subsystems, since:

$$\Lambda_c = \Lambda(C,A,B,R^{mxp}) = \bigcup_{i=1}^{N} \Lambda(C_i,A_i,B_i)$$

II.4.5. General Case of Systems Under Any Structural Stress

The notion of fixed modes may be generalized to any structural stresses (SEZ-8]a, LIN-83, PIC-84, SIL-82b REI-84a, etc.) (not necessarily decentralized stresses). In this case the control of a station i may depend on information from other subystems (fig. 2.2b).

Let us reformulate the problem by taking a look at system (2-4-1); the structural stress may be described in a convenient manner using a binary matrix (F with dimension mxp and element f_{ij}), such that $f_{ij} = 1$ if and only if the bond between the input of the j-th subsystem and the output of the i-th subsystem is permitted. Let us define for each input u_i the set of indices J_i with: /26

$$J_i = \{ j \text{ if } f_i = 11,\ i=1,\dots,m,\ j=1,\dots,p\} \qquad (2\text{-}4\text{-}15)$$

Therefore, the dynamic compensator defined in (2-4-5) becomes:

$$\begin{aligned} \overset{\circ}{z}_i(t) &= S_i\, z_i(t) + \sum_{j \in J_i} R_{ij}\, y_j(t) \\ u_i(t) &= Q_i\, z_i(t) + \sum_{j \in J_i} K_{ij}\, y_j + e_i(t) \end{aligned} \qquad (2\text{-}4\text{-}16)$$

The problem of stabilization consists of finding N controls of type (2-4-16) so that the closed loop system is asymptotically stable. To derive the stabilization conditions of the system, let us define the following set of constant control matrices:

$$K_F = \{K/K \in R^{m\times p},\ K_{ij} = 0 \text{ if } f_{ij} = 0\} \qquad (2\text{-}4\text{-}17)$$

By analogy with the case of a conventional decentralized control, the complementary subsystem of system (2-4-1) subjected to a type (2-4-17) type control is defined by:

Definition 2-4-7

(C_J, A, B_I) is a complementary subsystem of system (C,A,B) defined in (2-4-1) if I is an arbitrary subsystem of M indices with

$$\begin{aligned}
&M = \{1,\dots,m\} && (m = \text{number of inputs})\\
&M \supset I = \{ i_1,\dots, i_r\} \qquad r=1,\dots,m-1\\
&J = \bigcup_{i\in M-I} J_i = \{j_1,\dots,j_q\}\\
&B_I = \left[B_{i_1},\dots, B_{i_r}\right]\\
&C_J = \left[C_{j_1}^T,\dots, C_{j_q}^T\right]^T && (2\text{-}4\text{-}18)
\end{aligned}$$

Let us note that the definition of complementary subsystems for the case of a decentralized control (definition 2-4-3) is a special case of the above definition where subsets I and J are disjoined.

With this formulation, Sezer and Siljak (SEK-81a, SIL-82b) generalize the fixed mode definition (initially introduced for the Wang and Davison's decentralized control (WAN-73b)) for any structural stresses, as follows:

Definition 2-4-8

The fixed modes of the (2-4-1) system relative to the set of controls K_F defined in (2-4-17) are given by: /27

$$\Lambda_F\,(C,A,B,K_F) = \bigcap_{K \in K_F} \sigma\,(A+BKC)$$

In (SEZ-81a) (SIL-82b), it is shown that the stabilization and pole placement conditions in the present case remain the same as for the decentralized case, provided that the set Λ_F is considered in place of Λ in Wang's and Davison's findings (WAN-73b).

It is evident that if the set of matrices K_d, K_F are considered with $K_d \subset K_F \subset K_c$, then: $\Lambda_c \subset \Lambda_F \subset \Lambda_d \subset \sigma$

figure (2.6) depicts the inclusion ratio relationship between the different fixed mode sets according to the stress under consideration.

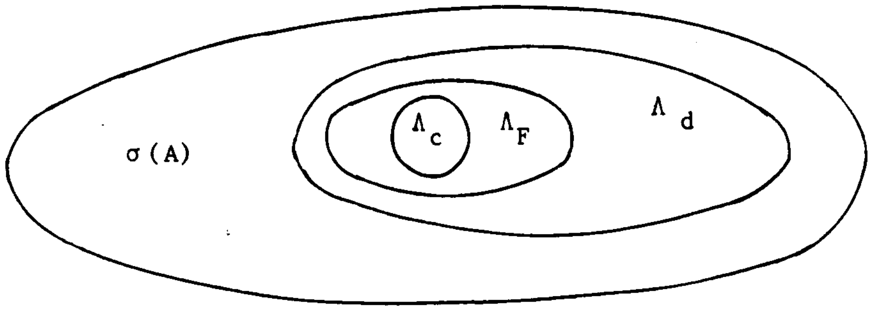


Fig. 2.6

II.5 CONCLUSION

In this chapter, are are interested in the problems of stabilization and pole placement using a dynamic control under structural stress. Several algorithms found in literature provide a solution to this problem (WAN-73b, SIL-78, SEZ-80, MAH80, SIN-81, sin-83, AND-84, etc.). However our goal here is simply to state the necessary and sufficient conditions for the existence of solutions. Let us note that with the exception of special cases (KUL-82), the problem of decentralized stabilization by static output feedback is not yet solved.

These results explicitly show the significance of the notion of fixed modes. Stabilization is possible if and only if the fixed modes are stable. Conversely, the existence of an unstable mode makes stabilization impossible. In this case, it will be necessary to change the control structure.

According to the algebraic definition of fixed modes and Wang's and Davison's results (WAN-73b), we clearly see that the fixed modes are modes that remain invariant for any value or nature (dynamic or static) of the control applied, because these modes are related to the control structure only. For this reason, in the following chapters, we will consider a static control alone /28 (without discussing the problem of decentralized observers) to present methods of fixed mode characterization and to indicate the possibilities of eliminating them or avoiding them if necessary.

CHAPTER III - ALGEBRAIC CHARACTERIZATION OF FIXED MODES /29

III.1 INTRODUCTION

In this chapter, after showing the importance of fixed modes in the previous chapter, we will discuss the methods of characterizing these modes in the time and frequency ranges.

Let us recall that fixed modes are system modes that are invariant to the control applied, and that the system class considered is the class of dynamic, linear, continuous, invariant in time, multivariable system with N control and observation systems described in the time range by equations (2-4-1) or globally by (2-4-2), and in the frequency range by (2-4-3) or /30 (2-4-4). These systems are assumed to be completely controllable and observable. This makes it possible to consider only fixed modes caused by structural stresses.

We have seen (chapter II) that the control dynamics plays no role in the existence of fixed modes. For this reason and to simplify the analysis, we will consider only the control class due to static output feedback (constant) in the overall form:

$$U(t) = K\,Y(t) + E(t) \qquad (3\text{-}1\text{-}1)$$

with $K \in K_d$ (2-4-8) (case of a decentralized control), or $K \in K_F$ (2-4-17) (case of a general structural stress), the vector $E(t)$ representing the system inputs.

If the control (3-1-1) is applied to the system (2-4-2), the closed loop system becomes:

$$\begin{aligned} \mathring{X}(t) &= (A + BKC)\ X(t) + B\ E(t) \\ &= D\ X(t) + B\ E(t) \end{aligned} \qquad (3\text{-}1\text{-}2)$$

where $D = A + BKC$ is the closed loop dynamic matrix.

III.2 FIXED MODE DETECTION

Two methods are available for detecting fixed modes:

-using their definition, proposed by Davison (DAV-76a),
-using their sensitivity, one of our proposals.

III.2.1. Fixed Mode Calculation Using Their Definition

Using the fixed mode definition (def. 2-4-2), Davison (DAV-76a) proposes the following algorithm.

Algorithm 3-2-1: (DAV-76a)

To find the fixed modes of system (2-4-2) with respect to the decentralized control K_d (2-4-8):

1) Calculate the natural values of A.
2) Arbitrarily select a matrix $K \in K_d$ (by pseudo-random number generation).
3) Calculate the natural values of the closed loop matrix $D = A+BKC$.
4) The fixed modes relative to K_d are, with a probability of 1, the common natural values of A and D.
5) If in doubt, arbitrarily select a new matrix $K \in Kd$ and go

to 3, if not go to the end.

Remark: a corresponding FORTRAN program is given in (TIT-86) /31

EXAMPLE 3-2-1 (DAV-83)

Given system (C,A,B) of (2-4-1) with three control stations (N=3) described by:

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -2 & \\ & & & -3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

By applying algorithm (3-1), for an arbitrary choice of K, i.e. K = diag. (0.13, - 0.17, - 0.1), we derive the following closed loop natural values:

σ (D) = {- 0.01023258, - 2.1597674, -2.1, - 3}

it is clear that s = -3 is a fixed mode relative to K.

The algorithm (3-1) may be used to actually calculate fixed modes of large systems [for example: steam generator of a ship, system of order 119 with three control stations (DAV-78a)].

III.2.2 Fixed Modes Calculation Using Their Sensitivity

In this section, we propose a new method of characterizing fixed modes under any structural stress. This method is based on the notion of the sensitivity of natural values of the closed loop system with parametric variations of the system.

Let us recall that the fixed modes are modes of the system that are invariant with respect to the set of control matrices. Let us redefine fixed modes in terms of mode sensitivity:

Definition 3-2-1 (TAR-84a)

The fixed modes of system (2-4-2) with respect to all control matrices K_F (2-4-17) are closed loop modes (3-1-2) that are insensitive to variations in non zero elements of the control matrix.

If the elements of a real matrix D change the value, the coefficients of its typical polynomial change as well. Therefore, the natural values also change position in the complex plan. Iif matrix D changes from dD, the variation of a simple natural value s_r, due to variation dD, is given by the following formula by Faddeev and Faddeeva (FAD-63):

$$d\ s_r = W_r^T .\ dD.\ V_r$$

where V_r and W_r are the respectively the natural values normalized to the right and to the left of matrix D and corresponding to s_r, i.e.:

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$$(D - s_r I)\ V_r = 0 \qquad (3\text{-}2\text{-}2)$$

$$W_r^T\ (D - s_r I) = 0 \qquad (3\text{-}2\text{-}3)$$

$$W_r^T\ V_r = 1 \qquad (3\text{-}2\text{-}4)$$

Based on relationship (3-2-1), it is shown (MOR-66) that:

$$d\ s_r = \frac{\text{Plotting }\{Q(s_r).dD}{\text{Plotting }\{Q(s_r\}} \qquad (3\text{-}2\text{-}5)$$

where Q(s) is the associated matrix of (sI-D), i.e. Q(s) = adj (sI-D). Relationship (3-2-5) is used to calculate the variation d s_r of a simple natural value corresponding to dD. Morgan (MOR-66) proposes an algorithm for calculating d s_r directly from relationship (3-2-5), Rosenbrock (ROS-65b) replaces Q(s) by its explicit expression available in (ROS-65a) and (GAM-66) and derives the following formula:

$$d\,s_r = \frac{\text{plot}\left\{\left\{\prod_{i(i\neq r)} (D-s_i I)\right\}.dD\right\}}{\prod_{i(i\neq r)} (s_r - s_i)} \qquad (3\text{-}2\text{-}6)$$

Rather than calculating the variation d s_r, the sensitivity of a natural value may be determined by calculating its gradient with respect to the variations of the elements d_{ij} of matrix D. Let us derive equation (3-2-2) with respect to d_{ij} (Lancaster (LAN-646) showed that for a simple natural value s_r the associated natural vectors V_r and W_r are continuous in d_{ij}); this gives us:

$$\left(\frac{dD}{d(d_{ij})} - \frac{d\,s_r}{d(d_{ij})} I\right) V_r + (D - s_r I) \frac{d\,V_r}{d(d_{ij})} = 0$$

by premultiplying by W^T and considering (3-2-3) and (3-2-4) we obtain:

$$\frac{d\,s_r}{d(d_{ij})} = W_r^T \frac{dD}{d(d_{ij})} V_r \qquad (3\text{-}2\text{-}7)$$

Let us remember that d is the closed loop matrix:

$$D = A + BKC = A + \sum_{i,j} b^i k_{ij} c_j \qquad (3\text{-}2\text{-}8)$$

where b_i and c_j are respectively the i-th column of B and the j-th line of C. Since we are assuming that the variation of s_r depends only on the variations of the elements k_{ij}, then (3-2-7) is expressed:

$$\frac{\partial s_r}{\partial k_{ij}} = W^T \frac{\partial D}{\partial k_{ij}} V_r$$

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relationship (3-2-8) gives: $\frac{\partial D}{\partial k_{ij}} = b^i c_j$

Finally, we obtain:

$$\frac{\partial s_r}{\partial k_{ij}} = W^T b^i c_j V_r \qquad (3\text{-}2\text{-}9)$$

If the control matrix undergoes a given structural stress, the elements $\frac{\partial s_r}{\partial k_{ij}}$ are zero for $k_{ij} = 0$. Therefore, the sensitivity matrix SK_r, with respect to a control under structural stress, which has a simple natural value s_r of the closed loop matrix, is given by:

$$SK_r = \left\| sk_{ij} \right\|_{\substack{i=1,\dots,m \\ j=1,\dots,p}} \tag{3-2-10}$$

with

$$sk_{ij} = \begin{cases} W^T\, b^i\, c_j\, V_r & \\ 0 & \text{si } k_{ij} = 0 \end{cases}$$

Proposition 3-2-1 (TAR84a)

Since system (3-1-2) with distinct modes, and the set of control matrices K_F (2-4-17), then s_r is a fixed mode of the system if and only if the sensitivity matrix SK_r relative to K_F, given in (3-2-10), is identically zero, or in an equivalent manner, if and only if:

$$S_r = \mathrm{Tr}\{ \prod_{i(i\neq r)} (D - s_i I) \, . \, dD \} = 0 \qquad i=1,\dots,n \tag{3-2-11}$$

with $D = A + BKC,\ K \in K_F,$

$dD = B.\ dK.\ C\ ,\ dK \in K_F$

Knowing that, for a simple natural value s_r, we have:

$$\mathrm{Tr}\ Q(s_r) = \prod_{i(i\neq r)} (s_r - s_i) \neq 0$$

then the demonstration of proposition 3-2-1 is direct according to definition (3-2-1) and relationships (3-2-6) and (3-2-10).

Using this proposition 3-2-1, we have the following algorithm for calculating fixed modes:

Algorithm 3-2-2 (TAR-84A)

To calculate the fixed modes of a system (that has simple modes) relative to the control set K_F:

1) Arbitrarily calculate a matrix $K \in K_F$ so that the modes of the closed loop system $\sigma(D) = \sigma(A+BKC)$ are simple.
2) $r = 1$.
3) Calculate for $s_r \in \sigma(D)$ the sensitivity matrix SK_r (3-2-10)
4) If SK_r is identically zero, then s_r is a fixed mode.
5) If $r < n$ perform $r = r+1$ and go to 3.
6) End.

Remark: A corresponding FORTRAN program is given in (TIT-86)

Example 3-2-2

Let us consider a globally controllable and observable system that has two control and observation systems described by:

$$\mathring{x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

$$y_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

$$y_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Let us apply an arbitrarily selected decentralized control, i.e.:

$K = \text{diag.}\ (k_1, k_2)\ \text{diag.}\ (1,5)$

Table (3.1) gives the results of applying algorithm (3-2-2) to this example.

Table (3.1)

s_r in B.F.	$SK_r = \dfrac{\partial s_r}{\partial K}$	$S_r = d\,s_r$ for $dK = K$
$-\,2,000000$	$\begin{bmatrix} -\,0,7645 & 0 \\ 0 & -\,0,1529 \end{bmatrix}$	$-\,29,999999$
$3,000000$	$\begin{bmatrix} 1,088 & 0 \\ 0 & 0,2177 \end{bmatrix}$	$19,999999$
$1,000000$	$\begin{bmatrix} -\,0,4953.10^{-15} & 0 \\ 0 & -\,0,3271.10^{-17} \end{bmatrix}$	$0,666138.10^{-14}$

Therefore for an accuracy $> 10^{-14}$ $s = 1$ is a fixed mode.

The algorithm was used to actually calculate the fixed modes of large systems [Ex: steam generator of a ship (chapter IV)].

It is difficult to generalize this approach to multiple mode systems. In effect, the variations of a multiple natural value sr of order q of a real matrix D, due to variations of elements dij of the matrix, are given by (PAR-74):

$$\frac{1}{q!}\left\{d^{q-1}\left[\mathrm{Tr}\ Q(s)\right]_{s=s_r}\right\}.\ d\,s_r = \mathrm{Tr}\left\{\sum_{k=1}^{q}\frac{1}{k!}\left[d^{k-1}\ Q(s)\right]_{s=s_r}.\ dD\right\} \qquad (3\text{-}2\text{-}12)$$ /35

where Q(s) = Adj(sI-D) which is an algebraic equation of order q. Therefore, generally speaking, a multiple natural value sr of multiplicity q, gives rise (after disturbance dD) to q simple natural values: sr +(dsr)1,..., sr+(dsr)i,...,sr+ (d sr)q, where (d sr)i i=1,..., q are the solutions of equation (3-2-12). Consequently, to have a fixed mode of order q we will have q conditions to satisfy, namely:

$$(d\ s_r)_i = 0 \qquad i=1,\ldots,q$$

and relationship (3-1-12) therefore gives:

$$S_r = \mathrm{Tr}\ \{ \sum_{k=1}^{q} \frac{1}{k!} \left[d^{k-1} \left[\mathrm{adj}(sI-D) \right] \right]_{s=s_r} .\ dD \} = 0 \qquad (3\text{-}2\text{-}13)$$

This expression is not of interest for calculating multiple fixed modes because it requires an analytical calculation of the associated matrix of (sI-D) and of its variations of order 1 to (q-1) due to variations of elements k_{ij} of K. Rather than calculating the variations (equation 3-2-13), the sensitivities of the multiple natural value (order q) of matrix D can be calculated by calculating the gradients of this natural value width respect to the elements k_{ij} of K; Lancaster (LAN-64) gives the following theorem:

Theorem 3-2-1 (LAN-64)

Let s be a multiple natural value of the real matrix D(R); where R is a parameter. Let us assume that the matrices of of the right V_q and left W_q natural vectors are selected so that $W_q V_q = I$. Therefore the q 1st derivatives of s with respect to R are the natural values of matrix $W_q D^* V_q$, where D^* is the 1st non zero derivative of matrix D with respect to R.

If we use the closed loop matrix for D and the elements k_{ij} of matrix K for R, we characterize the multiple fixed modes by:

Proposition 3-2-2

Given system (3-1-2) and the set of control matrices K_F, then s is a multiple fixed mode of order q of the system relative to K_F if and only if the natural values of matrices $W^T b_i c_j V_q$, i,j such that $K = \|k_{ij}\|\ K_F$, are all zero, W_q^T and V_q

---

* The conditions for the existence of matrices W_q and V_q are given in (LAN-64).

are the left and right natural vector matrices of D corresponding to s and selected so that $W_q^T \cdot V_q = I.$ /36

The demonstration of this proposal is direct according to theorem (3-2-1) and definition (3-2-1), since:

$$\frac{\partial D}{\partial k_{ij}} = \frac{\partial (A + \sum_{i,j} b^i \; k_{ij} \; c_j)}{\partial k_{ij}} = b^i \; c_j \neq 0$$

To use of this proposal to test whether s is a multiple fixed mode of order it is necessary to search for the matrices of the natural vectors W_q and V_q, and to calculate the natural values of n_k matrices of order q ($1 \leq n_k \leq mxp$, n_k is the number of non zero elements of matrix $K \in K_F$). This requires a long computer time and makes this method impractical.

III.2.3 Remarks

The two algorithms presented in this section are simple to use, and are generalizable for any control structure. They may also be used to calculate the centralized fixed modes (uncontrollable and/or unobservable modes). The only problem of using these algorithms on computer is to determine the real zero to calculate the sensitivity, and to decide when two natural values are equal for the Davison algorithm (problem of accuracy). Davison et al (DAV78-b) show that if a natural closed loop value s remains very similar to an open loop value, the transfer function calculated by considering that s is a fixed mode is a good approximation of the transfer system.

The two approaches cannot be used to have a physical interpretation of fixed modes. However, by calculating the fixed modes by sensitivity, as we will see in chapter IV, it is possible to know the origin and therefore the nature of such a mode.

III.3 ALGEBRAIC CHARACTERIZATION OF FIXED MODES IN THE TIME RANGE

We will present the characteristics in the time range (state space) by beginning with the oldest:

III.3.1. Characterization by Testing the Rank of a Matrix

Anderson and Clements (AND-81) give the following theorem:

Theorem 3-3-1 (AND-81a)

s is a decentralized fixed mode of system (2-4-1) if and only if at least one complementary subsystem exists (C_β, A, B_α) (def. 2-4-3) so that: /37

37.

$$\operatorname{rank} \begin{bmatrix} sI-A & B_\alpha \\ C_\beta & 0 \end{bmatrix} < n$$

It is evident that only the natural values of a should be tested. Theorem (3-3-1) is generalizable for any control structure. To calculate centralized fixed modes, it is simply (WIL-85) necessary to consider that sets and β are empty. For the case of any structural stress Pichai et al (PIC-84) proposed the following theorem:

Theorem 3-3-2 (PIC-84)

s is a fixed mode of system (2-4-1) relative to K_F (defined in 2-4-17) if a complementary subsystem exists (C_J, A, B_i) (def. 2-4-7) so that:

$$\operatorname{rank} \begin{bmatrix} sI-A & B_I \\ C_J & 0 \end{bmatrix} < n$$

This characterization is very interesting because it can be used to interpret the mechanism by which fixed modes appear in a system under any structural stress that is globally controllable and observable: one fixed mode is simultaneously uncontrollable by the set of (or I) stations and unobservable by the set of β (or J) stations. Moreover, by comparing the definition of transmission zeros of a system (see Appendix 1) with the test of theorems (3-3-1) or (3-3-2), it may be concluded that a necessary condition for s to be a fixed mode is that s must be a transmission zero of at least one complementary subsystem.

In this characterization it is necessary to calculate the rank of a generally complex matrix 2N-2 times, which is numerically not a well presented problem. To overcome this drawback and therefore to reduce the computer time required, Petel and Misra (PET-84) developed (for systems with an A matrix in Hessenberg's higher form (*)) a numerically equivalent condition to the rank test of theorem (3-3-1), according to which the fixed modes are transmission zeros of certain transfer functions, and therefore apply their computer algorithm of transmissions zeros (proposed in PET-84). For more details the reader is referred to (PET-84). Let us note that Petel and Misra's result mentioned above may be considered an extension of Davison and Ozguner's result (DAV-83) for diagonal systems (see III.3.4).

(*) For example Hessenberg's higher form for a system of order 4 is:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{14} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_4 \end{bmatrix}$$

Let us consider the linear system shown below, with 3a control and observation systems:

$$\overset{\circ}{X} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & b & 1 & 0 \\ 0 & 0 & 0 & a & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_3$$

$$y_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} X$$

$$y_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} X$$

$$y_3 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \end{bmatrix} X$$

The typical polynomial of the system is:

$$\Phi(s) = (s-b)\ (s-a)\ (s-3)\ (s^2-cs-1)$$

For a decentralized control Kd = diag (k11,k22,k33), the system has a fixed mode decentralized in s=b. In effect if the 1st and 2nd stations are considered together, we have:

$$\text{rank} \begin{bmatrix} bI-A & B_3 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix} = 4 < 5$$

If a=C, the system will have a fixed mode s=a, since:

$$\text{rank} \begin{bmatrix} aI-A & B_1\ B_3 \\ C_2 & 0\ \ 0 \end{bmatrix} = 4 < 5$$

Note that s=b is a transmission zero of (C1,C2,A,b3).

Recursive Characterization

Based on the above characterization, Davison and Ozguner (DAV-83) show that the fixed mode existence test of a system with N subsystems may be reduced to a fixed mode existence test of a system with N-1 subsystems. Consequently, to detect the fixed modes of a system with N control stations, it is simply necessary to study the fixed mode characterization for the two stations.

Theorem 3-3-3 (DAV-83)

I) s is not a decentralized fixed mode of the system (2-4-1) with N ≥ 3, if and only if s is not a fixed mode of any of the system with N-1 subsystems shown below.

$$(1)\qquad \left\{ \begin{bmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\ C_3 \\ \vdots \\ C_N \end{bmatrix}, A, \left[(B_1, B_2), B_3, \ldots, B_N \right] \right\}$$

$$(2)\qquad \left\{ \begin{bmatrix} C_1 \\ \begin{pmatrix} C_2 \\ C_3 \end{pmatrix} \\ \vdots \\ C_N \end{bmatrix}, A, \left[B_1, (B_2, B_3), \ldots, B_N \right] \right\}$$

$$\vdots$$

$$(N-2)\qquad \left\{ \begin{bmatrix} C_1 \\ \vdots \\ \begin{pmatrix} C_{N-2} \\ C_{N-1} \end{pmatrix} \\ C_N \end{bmatrix}, A, \left[B_1, \ldots, (B_{N-2}, B_{N-1}), B_N \right] \right\}$$

$$(N-1)\qquad \left\{ \begin{bmatrix} C_1 \\ \vdots \\ C_{N-2} \\ \begin{pmatrix} C_{N-1} \\ C_N \end{pmatrix} \end{bmatrix}, A, \left[B_1, \ldots, B_{N-2}, (B_{N-1}, B_N) \right] \right\}$$

II) s is not a decentralized fixed mode of the system (2-4-1), with N=2, if and only if the three following conditions are verified:

i) s is not a centralized fixed mode

ii) rank $\begin{bmatrix} sI-A & B_1 \\ C_2 & 0 \end{bmatrix} \geq n$

iii) rank $\begin{bmatrix} sI-A & B_2 \\ C_1 & 0 \end{bmatrix} \geq n$

This characterization does not provide anything new. In fact, it is the same as Andersin-Clements' previous characterization for part II. For part I of the theorem, for a system with N control stations we have to test the existence of fixed modes for N! systems and therefore N! tests (some of which are repeated). For example, for a system with N=4, we have to test 12 systems using Davison and Ozbuner's approach. Conversely, Anderson and Clement's approach is limited to an examination of only 7 systems (there are therefore 5 redundancies). In general, we have ($N! - 2^N - 2$) repeated tests. /40

III.3.3. Characterization by Common Transmission Zeros Between Certain Subsystems

III.3.3a. Characterization by Hu and Jiang (HUJ-84)

The system (2-4-1) is taken into consideration and the integers below are defined for i=1,...,N:

$$m_o = 0, \ \bar{m}_i = \sum_{j=o}^{i} m_j \ \& \ m = \bar{m}_N$$

$$p_o = 0, \ \bar{p}_i = \sum_{j=o}^{p} p_j \ \& \ p = \bar{p}_N$$

the system is expressed in the global form:

$$\overset{\circ}{X} = A\,X + B\,U$$
$$Y = C\,X$$

with $B = \left[B_1, B_2, \ldots, B_N\right] = \left[b_1, b_2, \ldots, b_m\right]$ or $B_i = \left[b_{\bar{m}_{i-1}+1}, \ldots, b_{\bar{m}_i}\right]$

$C = \left[C_1^T, \ldots\ldots, C_N^T\right]^T = \left[c_1^T, \ldots\ldots, c_p^T\right]^T$ or $C_i = \left[c_{\bar{p}_{i-1}+1}^T, \ldots, c_{\bar{p}_i}^T\right]^T$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} u^1 \\ u^2 \\ \vdots \\ u^m \end{bmatrix} \text{ with } u_i \begin{bmatrix} u^{\bar{m}_{i-1}+1} \\ \vdots \\ u^{\bar{m}_i} \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^p \end{bmatrix} \text{ with } y_i = \begin{bmatrix} y^{\bar{p}_{i-1}+1} \\ \vdots \\ y^{\bar{p}_i} \end{bmatrix}$$

It is assumed that the integers q_i and r_i are not negative so that:

$$0 \leqslant q_i \leqslant m_i \text{ and } \sum_{i=1}^{N} q_i > 0$$

$$0 \leqslant r_i \leqslant p_i \text{ and } \sum_{i=1}^{N} r_i > 0$$

For $q_i > 0$ and $r_i > 0$, the integers $f_{i,1}, \ldots, f_{i,q_i}$ and $g_{i,1}, \ldots, g_{i,r_i}$ are defined so that:

$$\bar{m}_{i-1}+1 \leqslant f_{i,1} < \ldots < f_{i,q_i} \leqslant \bar{m}_i$$
$$\bar{p}_{i-1}+1 \leqslant g_{i,1} < \ldots < g_{i,r_i} \leqslant \bar{p}_i$$

The following matrices are also defined: /41

$$\bar{B}_i = \left[b_{f_{i,1}}, b_{f_{i,2}}, \ldots, b_{f_{i,q_i}}\right] \longrightarrow \bar{B} = \left[\bar{B}_1, \bar{B}_2, \ldots, \bar{B}_N\right]$$
$$\bar{C}_i = \left[c_{g_{i,1}}^T, c_{g_{i,2}}^T, \ldots, c_{g_{i,r_i}}^T\right]^T \longrightarrow \bar{C} = \left[\bar{C}_1^T, \ldots\ldots, \bar{C}_N^T\right]^T \qquad (3\text{-}3\text{-}1)$$

(If $q_i = 0$ ($r_i = 0$) matrix $\bar{B}_i$,($\bar{C}_i$) disappears in 3-3-1).

Given the set of control matrices $\bar{K}_d$:

$$\bar{K}_d = \{K \;/\; K = \text{bloc diag } (K_i),\ K_i \in R^{q_i x r_i},\quad i=1,\ldots,N\} \tag{3-3-2}$$

and the system:

$$\begin{aligned} \dot{\bar{X}} &= A\,\bar{X} + \bar{B}\,\bar{U} \\ \bar{Y} &= \bar{C}\,\bar{X} \end{aligned} \tag{3-3-3}$$

with $\bar{B}$ and $\bar{C}$ given by (3-3-1), and

$$U = \begin{bmatrix} \bar{u}_1 \\ \vdots \\ \bar{u}_N \end{bmatrix} \text{ where } \bar{u}_i = \begin{bmatrix} u_{f_{i,1}} \\ \vdots \\ u_{f_{i,q_i}} \end{bmatrix}, \quad \bar{Y} = \begin{bmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_N \end{bmatrix} \text{ with } y_i = \begin{bmatrix} y_{g_{i,1}} \\ \vdots \\ y_{g_{i,r_i}} \end{bmatrix}$$

(if $q_i = 0$ ($r_i = 0$) the input $\bar{u}_i$ (output $\bar{y}_i$) disappears in (3-3-3).

Definition 3-3-1: (HUJ-84) Normal Subsystem

The system ($\bar{C}$, A, $\bar{B}$) defined in (3-3-3) under a decentralized control K E $\overline{Kd}$ (3-3-2) is called a normal subsystem ($\bar{C}$, A, $\bar{B}$; $\overline{Kd}$) of system (C, A, B; Kd). The set of normal subsystems is represented by SSN (C, A, B; Kd).

Definition 3-3-2: (HUJ-84) Nonsingular Normal Subsystem

If the dimensions of a normal subsystem are equal (i.e. if qi = ri, i=1,...,N), then the subsystem is called a normal and nonsingular subsystem. The set of normal and nonsingular subsystems is notated SSNN $(C, A, B; K_d)$.

Theorem 3-3-4: (HUJ-84)

s is a decentralized fixed mode of system (2-4-1) relative to Kd if and only if s is a fixed mode of all normal subsystems of the system relative to Kd, i.e.:

$$(C,A,B;K_d) = \bigcap_{(\bar{C},A,\bar{B};\bar{K}_d) \in SSN(C,A,B;K_d)} \Lambda\ (\bar{C},A,\bar{B};\bar{K}_d)$$

Theorem 3-3-5: (HUJ-84)

s is a decentralized fixed mode of the system (2-4-1) if and only if s is a common transmission zero (ZT) of all normal and nonsingular subsystems of the system, i.e.: /42

$$(C,A,B,K_d) = \bigcap_{(\bar{C},A,\bar{B};K_d) \in SSNN(C,A,B;K_d)} ZT(\bar{C},A,\bar{B})$$

These two theorems are more interesting in their negative forms. Considering the definition of transmission zeros (see Appendix 1), theorem 3-3-5 may be expressed:

Corollary 3-3-1 (HUJ-84)

s $\sigma(A)$ is not a decentralized fixed mode of system (2-4-1) if and only if a normal and nonsingular subsystem exists so that s is not a transmission zero of this subsystem, i.e.:

$$\text{rank} \begin{bmatrix} sI-A & \bar{B} \\ \bar{C} & 0 \end{bmatrix} = n + \sum_{i=1}^{N} r_i$$

III.3.3b. **Tarokh's Characterization** (TAR-84)

Let us consider a system with dimension n and m inputs and p outputs divided into N control stations, described by its overall state equation:

$$\overset{\circ}{X} = A X + B U \qquad U \in R^m \;\&\; X \in R^n$$
$$Y = C X \qquad Y \in R^p$$

Given C_q^i ($q=1,\ldots,qc$), the set of submatrices formed by i lines of matrix C, and B_r^j ($r=1,\ldots,.\ r_s$) the set of submatrices formed by j columns of matrix B with

$$r_b = \frac{m!}{(m-i)!\ i!} \quad \text{and} \quad q_c = \frac{p!}{(p-i)!\ i!}$$

It is said that a subsystem has dimension i if a has the same number of inputs and outputs, i.e. i; it will be notated:

$$(C_q^i,\ A,\ B_r^i) \quad \begin{array}{l} i = 1,\ldots,\ \min\ (m,p) \\ q = 1,\ldots,\ q_c \\ r = 1,\ldots,\ r_b \end{array}$$

Theorem (3-3-6) (TAR-84)

One necessary and adequate condition for s to be a fixed mode relative to K_d is:

$$\text{rank} \begin{bmatrix} sI-A & B_r^i \\ C_q^i & 0 \end{bmatrix} < n+i$$

where $(C_q^i,\ A,\ B_r^i)$ are subsystems.

of the system corresponding to structurally nonsingular matrices of K_d^T.

Using the transmission zero definition (see Appendix 1) this /43
theorem is reformulated as follows:

Corollary 3-3-2

A necessary and adequate condition for s not be to a decentralized fixed mode is that s is not a transmission zero of any subsystem corresponding to a structurally nonsingular submatrix of K_d^T .

The findings are exactly the same as those of Hu and Jiang. It is concluded that normal and nonsingular subsystems are subsystems which correspond to nonsingular submatrices of K .

Note that if m=p, then a necessary condition for the system to have a fixed mode in s is that s is a transmission zero of the system.

It should be stressed that Hu's and Jiang's and Tarokh's characterizations are based on the expansion of a typical polynomial of the closed loop system in terms of K submatrices and the transfer matrix. This expansion is used for the first time to characterize the fixed modes in the frequency range by Seraji (SER-82) and Vidasagar and Viswanadham (VID-82) (see next section).

Example 3-3-2

Let us take example 3-3-1. The subsystems corresponding to nonsingular matrices of K_d^T are:

with dimension i=1 (C_1,A,B_1), (C_2,A,B_2) and
(C_3,A,B_3)
with dimension i=2: (C_1C_2,A,B_1B_2),
(C_1,C_3,A,B_1B_3)
and (C_2C_3,A,B_2B_3)
with dimension i=3: $(C_1C_2C_3,A,B_1B_2B_3)$

it is easy to check whether s=b is a common transmission zero of all these subsystems, therefore s=b is a decentralized fixed mode of the system. Note that the number of subsystems to be tested is 7, and that the number of complementary subsystems is 6. In this case, Anderson's and Clement's characterization requires fewer tests and therefore less calculations than that of Hu and Jiang or Tarokh. In effect, the number of normal and nonsingular subsystems depends on the number of nonzero elements of matrix K, and the number of these subsystems decreases as matrix K becomes hollower.

III.3.4. Special Case of Diagonal Systems

The following system, which has two control and observation systems, is taken into consideration:

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$$\overset{\circ}{X} = \left[\begin{array}{c:ccc} s_1 & & \mathcal{O} & \\ \hdashline \theta & s_2 & & \\ & & \ddots & \\ & & & s_n \end{array}\right] X + \left[\begin{array}{c} B_1^* \\ \hdashline \\ \bar{B}_1 \end{array}\right] u_1 + \left[\begin{array}{c} B_2^* \\ \hdashline \\ \bar{B}_2 \end{array}\right] u_2$$

$$y_1 = \left[\begin{array}{c:c} C_1^* & \bar{C}_1 \end{array}\right] X \qquad (3\text{-}3\text{-}4a)$$

$$y_2 = \left[\begin{array}{c:c} C_2^* & \bar{C}_2 \end{array}\right] X$$

where $u_i \in R^{m_i}$ and $y_i \in R^{p_i}$ and $s_i \in C$, B_1^* (C^*008_1) and B_2^*) (C_2^*) are respectively lines (columns) of dimension m_1 and m_2 (p_1 and p_2). i.e.:

$$\bar{B}_1 \triangleq \left[b_1^1, b_2^2, \ldots, b_{m_1}^1\right] \quad \text{and} \quad \bar{B}_2 \triangleq \left[b_1^2, b_2^2, \ldots, b_{m_2}^2\right] \qquad (3\text{-}3\text{-}4b)$$

$$\bar{C}_1 \triangleq \begin{bmatrix} c_1^1 \\ \\ c_{p_2}^1 \end{bmatrix} \quad \text{and} \quad \bar{C}_2 \triangleq \begin{bmatrix} c_1^2 \\ \\ c_{p_2}^2 \end{bmatrix}$$

Davison and Ozguner (DAV-83), assuming that the s_i $i=1,...,n$, are distinct, give the following theorem:

Theorem 3-3-7 (DAV-83)

s_i is not a decentralized fixed mode of system (3-3-4) if and only if:

i) s1 is not a centralized fixed mode, i.e. $(B_1^* B_2^*) \neq 0$ & $\begin{pmatrix} C_1^* \\ C_2^* \end{pmatrix} \neq 0$

ii) condition $B_2^* = 0$ and $C_r^* = 0$ and s_1 is a transmission zero of:

$$\left\{ C_i^r, \begin{bmatrix} s_2 & & \\ & \ddots & \\ & & s_n \end{bmatrix}, b_j^q \right\} \quad \begin{matrix} \forall\ i \in \{1,2,\ldots,p_r\} \\ \forall\ i \in \{1,2,\ldots,m_q\} \end{matrix}$$

is not verified for $q = 1,2$ and $r = 1,2$ and $q \neq r$.

This theorem is easily generalizable, for a system with N: control stations by using theorem (3-3-3) (see PET-84).

Note that the controllability and observability of s_1, for a centralized control, depends only on B_1^*, B_2^*, C_2^* and are independent of matrices $\overline{B_1}$, $\overline{B_2}$, $\overline{C_2}$ and $\overline{C_2}$ and of the natural values $s_1,...,s_n$. In the case of a decentralized control, this independence is no longer assured.

III.3.5. Special Case of Interconnected Systems /45

The systems under consideration in this section are systems made up of systems that are interconnected statically and described by:

$$\overset{\circ}{X}_i = A_{ii} X_i + \sum_{\substack{j=1 \\ i\neq j}}^{N} A_{ij} X_j + B_i u_i$$

$$y_i = C_i X_i \qquad (3\text{-}3\text{-}5)$$

where $X_i \in R^{n_i}$, $u_i \in R^{m_i}$ and $y \in R^{p_i}$, matrices A_{ij} being constant matrices of the appropriate dimensions.

III.3.5a Characterization with Stress on the Interconnections

For the interconnection class expressed:

$$A_{ij} \triangleq L_{ij} K_{ij} M_{ij} \quad i,j=1,\ldots,N \qquad (3\text{-}3\text{-}6)$$

where the matrices K_{ij} are interconnection gains and L_{ij}, M_{ij} are arbitrary matrices, Davison (DAV-76b) gives the following theorem:

Theorem 3-3-8 (DAV-76b)

Since system (3-3-5) is controllable and locally observable (i.e. systems (C_i, A_{ii}, B_i) i=1,1..., N are controllable and observable) then system (3-3-5) (3-3-6) does not have fixed modes for virtually all interconnections K_{ij} i,j=11,...,N and $i\neq j$. (Namely that the nonzero gain class K_{ij} for which (3-3-5) has fixed modes is void, i.e. reduced to a general hypersurface of the space for parameters with nonzero elements of K_{ij} (DAV-76b).

The most interesting result is given for the interconnection class in the form:

$$A_{ij} = B_i K_{ij} C_j \quad i,j=1,\ldots,N \text{ and } i\neq j \qquad (3\text{-}3\text{-}7)$$

by Saeks (SAE-79) and Davison (DAV-79)

Theorem 3-3-9

A necessary and adequate condition for system (3-3-5) and (3-3-7) not to have fixed modes is that the local subystems are controllable and observable, i.e (C_i, A_{ii}, B_i) $i=1,\ldots,$ controllable and observable N.

Saeks (SAE-79) proposes the result below for this system class, showing that decentralized fixed modes are centralized fixed modes, given by the union of uncontrollable and/or unobservable modes of each subsystem:

Theorem 3-3-10 (SAE-79) /46

The fixed modes of system (3-3-5) and (3-3-7) are given by:

$$\Lambda(C,A,B;K_d) = \Lambda_c(C,A,B;R^{mxp}) = \bigcup_{i=1}^{N} \Lambda(C_i,A_{ii},B_i)$$

Interesting results are found in (DAV-83) for a control by decentralized state feedback ($C_i=I$), for this system class.

III.3.5b. Characterization Using Block Diagonally Dominant Matrices (ARM-82)

System (3-3-5) is globally considered in the form:

$$\begin{aligned} \overset{\circ}{X} &= A\,X + B\,U \\ Y &= C\,X \end{aligned} \qquad (3\text{-}3\text{-}8)$$

with $A = \{A_{ij},\ i,j=1,\ldots,N\} \in R^{nxn}$ $\quad n = \sum_i n_i$

$B =$ block.diag. $(B_1,\ldots,B_N) \in R^{nxm}$ $\quad m = \sum_i m_i$

$C =$ block.diag. $(C_1,\ldots,C_N) \in R^{pxn}$ $\quad p = \sum_i p_i$

If a decentralized control is applied in the following form:

$$u_i = K_{ii} \quad y_i \qquad (3\text{-}3\text{-}9)$$

the closed loop dynamic matrix D becomes:

$$D = A+BKC = \begin{bmatrix} \hat{A}_{11} & A_{12} & \cdots\cdots & A_{1n} \\ A_{21} & \hat{A}_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n1} & \cdots\cdots & \cdots & \hat{A}_{nn} \end{bmatrix} \quad \text{with } \hat{A}_{ii} = A_{ii}+B_iK_{ii}C_i$$

if the diagonal matrices A_{ii} are not singular and if:

$$\| \hat{A}_{ii}^{-1} \| < \sum_{\substack{j=1 \\ i \neq j}}^{N} \| A_{ij} \| \quad \text{for} \quad i=1,\ldots,N$$

where $\|A\|$ is a norm of matrix A, defined by:

$$\|A\| = \max_{i} \sum_{j} |a_{ij}|$$

then matrix D is block diagonally dominant.

Theorem 3-3-11 (FEI-62)

If matrix D = A+BKC is block diagonally dominant, then it is nonsingular.

Using theorem (3-3-11) Armentano and Singh (ARM-82) characterize decentralized fixed modes by the following corollary:

Corollary 3-3-3 (ARM-82)

If s is a decentralized fixed mode of system (3-3-5), then:

$$\| (\hat{A}_{ii} - s_i I_i)^{-1} \|^{-1} < \sum_{\substack{j=1 \\ j \neq i}}^{N} \| A_{ij} \| \quad \forall\, K_{ii} \in R^{m_i x p_i} \qquad (3\text{-}3\text{-}10)$$

is verified at least for i, $i \in \{1,\ldots,N\}$.

This characterization is interesting because it enables

Armentano and Singh to develop a method for selecting a control structure eliminating the fixed modes (ARM-82) (see Ch. VII). An example for applying this method of characterization is given in chapter VII.

III.3.6 Interpretation

Anderson's and Clements' characterization (AND-81a) reveals that the existence of fixed modes of a system with N control stations signifies the existence of fixed modes for a system with two aggregated stations (control and observation). Davison's and Ozguners' (DAV-83) recursive characterization coincides with Anderson's and Clement's (AND-81a) characterization, but it also proposes more systematically a description of all system partitions (with repetition of certain partitions).

Let us consider a station partition into two aggregated stations, and let us use the transmission zeros definition (DAV-74)(DAV-74) (Appendix 1). Characterizations by Davison and Ozguner (DAV-83) and by Anderson and Clements (AND-81a) show that in order for $s \in \sigma$ (1A) not to be a fixed mode of the system:

i) s must be controllable by station α(or β)
ii) s must be observable by station β (or α)
iii) s must not be a transmission zero of one of the complementary subsystems (C_β, A, B_α) and (C_α, A, B_β).

A very interesting interpretation of conditions i) and ii) is that, for the partition under consideration, s is a fixed mode if it is simultaneously uncontrollable by one of the aggregated stations, either α, and unobservable by the other, or β. Figure 3.1 illustrated this interpretation.

According to this interpretation, a sufficient condition not /48
to have a fixed mode in s is that s is controllable and observable by station j (i.e. j such as subsystem (C_j, A, B_j) is controllable and observable) which may correspond to a corollary

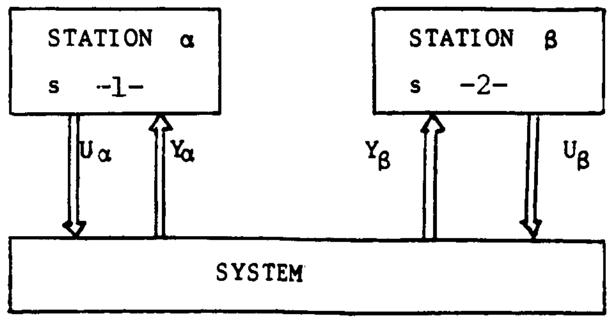


Fig. 3.1 - Characterization of Fixed Mode s

Key: 1-Uncontrollable; 2-Unobservable.

of theorem (3-3-4) by Hu and Jiang. In short, Anderson's and Clement's characterization (AND-81a) may be derived from Hu's and Jiang's results (HUJ-8a4).

Condition iii) provides a second interesting interpretation of fixed modes, namely: a necessary condition for a mode to be fixed is that it is a transmission zero of at least one complementary subsystem. Furthermore, results by Hu and Jiang and by Tarokh show that a fixed mode is a common transmission zero of normal and nonsingular subsystems (subsystems corresponding to structurally nonsingular submatrices of K_d^T), which is used to reformulate the necessary condition above as follows: if is a fixed mode,then s is a transmission zero of all complementary subsystems belonging to the set of normal and nonsingular subsystems. In short, a fixed mode is sort of a transmission zero. Moreover, a transmission zero of a single input/single output system corresponds to the frequency which cancels the numerator of the transfer function. It may be concluded that a fixed mode corresponds to the frequency that interrupts the information flow between inputs and outputs of certain subsystems.

Although the characterizations presented here are highly interesting for interpreting fixed modes in terms of uncontrollable and unobservable modes, or transmission zeros, and despite the presence of efficient algorithms for calculating transmission zeros, these characterizations remain impractical for calculating the fixed modes of a system because of the long computer time required.

Finally, the correspondence between Wang's and Davison's results (WAN-73b) land those of Corfmat and Morse (COR-76b) presented in the preceding chapter is now clear, because Corfmat's and Morse's result establishes transmission zeros through the "remnant polynomial", and the fixed modes are, so to speak, transmission zeros.

III.4 ALGEBRAIC CHARACTERIZATION OF FIXED MODES IN THE FREQUENCY RANGE /49

In this section, we will be interested in characterizations in the frequency range. Let us recall that in this range a system is described by an input-output (U - Y) relationship which may be presented in the form of "Matrix fraction description"" polynomial matrices, or by a rational transfer matrix (2-4-3) or (2-2-4).

III.4.1. Characterization by Testing the Rank of a Polynomial Matrix

The system considered here is described by its "left matrix fraction description".

$$Y(s) = S^{-1}(s)\ T(s)\ U(s)$$
$$\text{or} \quad S(s)\ Y(s) = T(s)\ U(s) \qquad (3\text{-}4\text{-}1)$$

with S(s) and T(s), for a system with N control stations partitioned as follows:

$$S(s) = [S_1(s), \ldots, S_N(s)]$$
$$T(s) = [T_1(s), \ldots, T_N(s)]$$

$S_i(s)$ and $T_i(s)$ are polynomial matrices of appropriate sizes. The subsystem i has m_i inputs and p_i outputs, therefore $S_i(s)$ and $T_i(s)$ have p_i and m_i columns respectively. Anderson and Clements (AND-81a) characterize fixed modes using the following theorem:

Theorem 3-4-1 (AND-81a)

System (3-4-1) has a fixed mode relative to the set of decentralized controls (K: block diagonal) in s_o if and only if an unempty subset exists $\{i_i,\ldots, i_j\}$ of $\{1,\ldots, N\}$ for which

$$\operatorname{rank}\left[S_{i_1}(s_o),\ldots, S_{i_j}(s_o), T_{i_1}(s_o),\ldots, T_{i_j}(s_o)\right] < \sum_{i=1}^{j} p_i$$

Recently Zheng (ZHE-84) came to this same theorem with a different demonstration.

III.4.2. Characterization by Common Transmission Zeros Between Certain Subsystems

Let us consider system (2-4-4), with N control stations, described by its transfer function:

$$G(s) = \frac{N(s)}{\phi(s)} = \frac{C\ \mathrm{Adj}(sI-A)\ B}{\det(sI-A)}$$

If we suppose that an output feedback control $U = K\ Y + E$ is applied to the system, then the closed loop transfer matrix is: /50

$$H(s) = (I-GK)^{-1}G = \frac{\mathrm{adj}(I-GK)\ N(s)}{\phi(s)\ \det(I-GK)}$$

therefore the typical closed loop polynomial is expressed:

$$\Phi(s,K) = \det(sI-A-BKC) = \Phi(s).\det(I-GK) \qquad (3\text{-}4\text{-}2)$$

Let the number of elements of set I be represented by $\|I\|$. If $\|I\| = \|J\|$ then $X\left[{}^{I}_{J}\right]$ represent the minor of matrix X formed by the elements of lines I and of columns J. By expanding det(I-GK) according to the principal minors of GK, expression (3-4-2) is expressed:

$$\Phi(s,K) = \phi(s)\left[1 + \sum_{i=1}^{p} \sum_{\|I\|=i} (-G(s)K)\left[{}^{I}_{J}\right]\right]$$

using Binet-Cauchy's formula (GAM-66), we obtain:

$$\Phi(s,K) = \phi(s)\left[1 + \sum_{i=1}^{p} \sum_{\|I\|=i} \sum_{\|J\|=i} (-1)^{i}\, G\left[{}^{I}_{J}\right] K\left[{}^{J}_{I}\right]\right]$$

where $G[{}^{I}_{J}]$ is a the minor of G(s) formed by elements of lines I and columns J (this minor represents a subsystem of dimension $\|I\|$, see III.3.3b).
$K[{}^{I}_{J}]$ is defined in a similar manner from K.
This gives us the typical closed loop polynomial:

$$\Phi(s,K) = \phi(s) + \sum_{i=1}^{p} \sum_{\|I\|=i} \sum_{\|J\|=i} (-1)^{i}\, Z\left[{}^{I}_{J}\right] K\left[{}^{J}_{I}\right] \qquad (3\text{-}4\text{-}3)$$

with $Z\left[{}^{I}_{J}\right] = \phi(s)\, G\left[{}^{I}_{J}\right]$

Note that $Z[{}^{I}_{J}]$ is a polynomial for every I and J, and that its roots are transmission zeros of the subsystem represented by the minor $G[{}^{I}_{1J}]$; we actually have:

$$Z\left[{}^{I}_{J}\right] = \begin{vmatrix} sI-A & B_J \\ C_I & 0 \end{vmatrix} = |sI-A| \,.\, |C_I(sI-A)^{-1} B_J| = \phi(s)\, G\left[{}^{I}_{J}\right]$$

Note also that formula (2-4-3) is valid for any feedback matrix structure K.

III.4.2a. Seraji's Characterization: (SER-82)

Seraji (SER-82) considers the system composed of N single input/single output subsystems, i.e.: $m_i = p_i = 1$, i=-1,...,N, i.e. m = p = N. The decentralized control matrix is therefore diagonal FK = diag(k_i). If I = F{i,..., iq] a subset of {1,...,m} with $i_i < i2 < iq$ then expression (2-4-3) is expressed:

$$\Phi(s,K) = \phi(s) + \sum_{q=1}^{m} \sum_{||I|| = q} k_{i_1} \times k_{i_2} \times \ldots \times k_{i_q} \times Z\left[{I \atop I}\right] \qquad (3\text{-}4\text{-}4)$$

based on this expression Seraji (SER-82) demonstrated the following theorem:

Theorem 3-4-2 (SER-82) /51

A necessary and sufficient condition for s to be a decentralized fixed mode of system (2-4-3), composed of single input single output subsystems, is that s is a transmission zero common to all subsystems of dimension j=1,...,N formed by selecting the same inputs and outputs of the system.

This theorem may be reformulated by: fixed modes of the system are zeros of the highest common divider of all $Z{I \atop I}]$ and $\Phi(s)$.

When the local controls are not scalar ($m_i \neq p_i$, $m_i \geq 1$ and $p_i \geq 1 \rightarrow$ block diagonal K matrix) Seraji (SER-82) converts the converted system (diagonal control matrix in the new base). If P is the conversion matrix:

$$K = P\hat{K} \quad \text{with } \hat{K} = \text{diag}(k_1, \ldots, k_p)$$
$$\hat{G} = G\,P$$
$$P = K\,\text{diag}\left(\frac{1}{k_1}, \ldots, \frac{1}{k_p}\right)$$

The fixed modes of the system are fixed modes of the converted system which are given by theorem (3-4-2) applied to this converted system.

Seraji's approach is interesting for this single input and single output case, but it seems difficult to use in a general case. Seraji (SER-82) provides the necessary and sufficient condition for the existence of fixed modes of the converted system, but in the original base, these conditions are not necessary.

III.4.2b. Characterization by Vidyasagar and Vioswandham (VID-83)

Vidyasagar and Vioswanadham consider the traditional case of a decentralized control where K is a block diagonal matrix with arbitrary diagonal blocks. Some K minors are therefore identically zero, and the corresponding terms do not appear in expansion (3-4-3). According to (3-4-3) a fixed polynomial is p.g.c.d. of $\Phi(s)$ and $Z[^I_J]$ which correspond to $K[^J_I]$ which structurally are not zero.

Vidyasagar and Viswanadham (VID-82 and 83) characterize a decentralized fixed polynomial by the following theorem:

Theorem 3-4-3 (VID-82 and 83)

The fixed polynomial of system (2-4-3) relative to the decentralized control (K: block diag. $(K_1,\ldots, K_N)$, $K_i \in R^{m_i \times p_i})$ is given by:

$$\Psi(s) = \text{p.g.c.d.}\left\{ \phi(s)\ ;\ Z\begin{bmatrix} I_{i_1} \cup I_{i_2} \cup \cdots \cup I_{i_r} \\ J_{i_1} \cup J_{i_2} \cup \cdots \cup J_{i_r} \end{bmatrix} \right\}$$

where $\{i_1,\ldots,i_r\}$ is a subset of $\{1,\ldots,N\}$, and /52

$I_{i_q} \subset \mathbf{P}_{i_q}$, $J_{i_q} \subset \mathbf{M}_{i_q}$ with $\|I_{i_q}\| = \|J_{i_q}\|$ for every q and

$$\mathbf{P}_1 = \{1,\ldots, p_1\}$$

$$\mathbf{P}_2 = \{p_1+1,\ldots, p_1+p_2\}$$

$$\vdots$$

$$\mathbf{P}_N = \{(\sum_{i=1}^{N-1} p_i)+1,\ldots, \sum_{i=1}^{N} p_i\}$$

The M_i are defined in a similar manner by replacing the p_i by the m_i.

If the subsystems are single input/single output, theorem (3-4-3) is reformulated by the following corollary which is equivalent to Seraji's results presented in the previous chapter.

Corollary 3-4-1 (VID-83)

If $m = p = N$ and $m_i = p_i = 1$ then the decentralized fixed polynomial is given by:

$$\Psi(s) = \text{p.g.c.d.}\{\phi(s);\ Z\left[{}^{I}_{J}\right],\ I \subset \{1,\ldots,m\}\}.$$

Note that the characterization presented here may be formulated in terms of polynomial matrices ("Coprime factorization") i.e.: $G(s) = S_{-1}(s)\ T(s)$. (For more details see VID-83).

II.4.2C. Characterization By Hu and Jiang (HUJ-84)

By considering the notations of sections III.3.3a, the results by Hu and Jiang (HUG-84) appear in the frequency range as follows:

Theorem 3-4-4 (HUG-84)

The decentralized fixed modes of system (2-4-3) are given by:

$$(G(s);K_d) = \bigcap_{(\bar{G}(s);\bar{K}_d) \in \text{SSN}\ (G(s);K_d)} \Lambda\ (\bar{G}(s);\bar{K}_d)$$

where $\overline{G(s)}$ is the transfer matrix of a normal subsystem (def. 3-3-1) and $\overline{K_d}$ is the control set defined in (3-3-2).

Theorem 3-4-5 (HUJ-84)

The decentralized fixed modes of system (2-4-3) are given by:

$$(\bar{G}(s);\bar{K}_d) = \bigcap_{(\bar{G}(s);\bar{K}_d) \in \text{SSNN}\ (G(s);K_d)} \text{Z.T.}\ (\bar{G}(s))$$

where $ZT(X)$ are transmission zeros of X.

This theorem is more interesting in its negative form, i.e.:

Corollary 3-4-2 (HUG-84)

s_0 is not a decentralized fixed mode of system (2-4-3) if a normal and nonsingular subsystem exists that does not have a transmission zero in s_0, i.e.: $\Phi(s_0).\det G(s_0) \neq 0$

III.4.2d. Taraokh's Characterization (TAR-84)

Tarokh's results (TAR-84) are expressed in the frequency range as follows:

Theorem 3-4-6 (TAR-84)

s_0 is a decentralized fixed mode of system (2-4-3) if and only if it is a transmission zero of all subsystems, whose dimension is i i=1,...,min (m,p), of G(s) corresponding to nonsingular submatrices of K_d^T.

The four characterizations of this section are all based on the expansion of the typical closed loop polynomial (3-4-2). They use different notations, but they may easily be reduced to each other. Nevertheless, Vidyasagar and Viswanadhams' characterization is the easiest to use because it is systematic in determining the subsystems to be considered.

Since formula (3-4-3) is valid for any structure of matrix K under consideration, it may be concluded that the characterizations of this section are generalizable to any control structure. In particular the case of a centralized control, Vidyasagar and Viswanadham's theorem (3-4-3) may be expressed in the form of the following lemma:

Lemma 3-4-1 (VID-83)

The centralized fixed polynomial of system (2-4-4) is given by: $\Psi_c(s) = \text{p.g.c.d.}\{\phi(s);\ Z\left[{}^{I}_{J}\right],\ I \subset \{1,\dots,m\}\ ,\ J \subset \{1,\dots,p\}\}$

In the case of a control with any structure, Tarokh's theorem (3-4-6) is valid after replacing K_d by K_F where K_F represents the structure of the control under consideration.

Example 3-4-1

Let us take another look at example (3-3-1), the transfer matrix of the system is given by:

$$G(s) = \left[\begin{array}{c|c|c} 0 & \frac{1}{s-a} & 0 \\ \hline \frac{s-c}{s^2-cs-1} & 0 & 0 \\ \hline 0 & \frac{1}{(s-b)(s-a)} & \frac{1}{s-3} \end{array}\right]$$

For a totally decentralized control, the fixed polynomial of the system is given, according to theorem (3-4-3) by: /54

$$\Psi(s) = \text{p.g.c.d.}\{\,\phi(s),\ z\begin{bmatrix}1\\1\end{bmatrix},\ z\begin{bmatrix}2\\2\end{bmatrix},\ z\begin{bmatrix}3\\3\end{bmatrix},\ z\begin{bmatrix}12\\12\end{bmatrix},\ z\begin{bmatrix}13\\13\end{bmatrix}, z\begin{bmatrix}23\\23\end{bmatrix},\ z\begin{bmatrix}123\\123\end{bmatrix}\}$$

we have:

$$\left.\begin{aligned} &\phi(s) = (s-b)\ (s-a)\ (s-3)\ (s^2-cs-1) \\ &z\begin{bmatrix}1\\1\end{bmatrix} = z\begin{bmatrix}2\\2\end{bmatrix} = z\begin{bmatrix}13\\13\end{bmatrix} = z\begin{bmatrix}23\\23\end{bmatrix} = 0 \\ &z\begin{bmatrix}3\\3\end{bmatrix} = (s^2-cs-1)\ (s-a)\ (s-b) \\ &z\begin{bmatrix}12\\12\end{bmatrix} = -\ (s-c)\ (s-b)\ (s-3) \\ &z\begin{bmatrix}123\\123\end{bmatrix} = -\ (s-c)\ (s-b) \end{aligned}\right\}\quad \Psi(s) = s-b$$

however if a=c then $\Psi(s) = (s-b)\ (s-a)$.

Now let us consider the following control matrix:

$$K_F = \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & k_{23} \\ 0 & 0 & k_{33} \end{bmatrix}$$

element k_{23} connects the 3rd subsystem to the 2nd subsystem, by combining these two subsystems into a single one with $k_{32} = 0$. by applying theorem (3-4-3), we have:

$$\Psi(s) = \text{p.g.c.d.}\{\ \phi(s),\ Z\begin{bmatrix}1\\1\end{bmatrix},\ Z\begin{bmatrix}2\\2\end{bmatrix};\ Z\begin{bmatrix}3\\2\end{bmatrix};\ Z\begin{bmatrix}3\\3\end{bmatrix},\ Z\begin{bmatrix}12\\12\end{bmatrix};\ Z\begin{bmatrix}13\\12\end{bmatrix};\ Z\begin{bmatrix}13\\13\end{bmatrix},\ Z\begin{bmatrix}23\\23\end{bmatrix},\ Z\begin{bmatrix}123\\123\end{bmatrix}\}$$

Moreover $Z[{}^{3}_{2}] = (s^2-cs-1)\ (s-3)$, and the system no longer has a fixed mode, unless if a=3 or b=3; it has a simple fixed mode if a=b=3.

III.4.3 Characterizations of the Closed Loop Transfer Matrix By Zeros

In this section, we give a necessary condition for the existence of fixed modes in terms of a closed loop transfer matrix.

Theorem 3-4-7 (BIN-78) (BER-81)

The matrix derived from the typical polynomial $\Phi(s,K)$ of the closed loop system (C,A,B,) relative to control K is given by:

$$\frac{\partial\,\Phi(s,K)}{\partial K} = -\ M^T(s,K)$$

where M(s,K) is the numerator of the closed loop transfer matrix, i.e.: $M(s,K) = H(s,K)\ \Phi(s,K) = C\ \mathrm{Adj}(sI-A-BKC)\ B$.

Proposition 3-4-1

A necessary condition in order for s_0 to be a fixed mode of the system with respect to the set of controls K_F is that the projection of the numerator of the closed loop transfer matrix to K_F^T is zero for $s=s_0$, i.e.

$M(S_0,K_F) = 0$

Demonstration /55

Theorem (3-4-7) expresses that the derivative of $\Phi(s,K_F)$

relative to the set of controls under structural stress K is given by projecting the numerator of the closed loop transfer matrix to K_F^T, i.e.:

with

$$\frac{\partial \Phi(s,K_F)}{\partial K_F} = - M(s,K_F) = \|m_{ij}\| \begin{matrix} i=1,\dots,m \\ j=1,\dots,p \end{matrix} \qquad (3-4-5)$$

$$m_{ij} = \begin{cases} m_{ji} \\ 0 & \text{if } k_{ij} = 0 \end{cases}$$

where the m_{ij} are elements of the numerator of the closed loop transfer matrix.

Furthermore, if the system has a fixed mode then the typical closed loop polynomial divides the fixed polynomial, it is expressed: $\Phi(s,K_F) = \Psi(s).\ P(s,K_F)$

The derivative of $\Phi(s,K_F)$ with respect to K_F is therefore:

$$\frac{\partial \Phi(s,K_F)}{\partial K_F} = \Psi(s) \frac{\partial P(s,K_F)}{\partial K_F}$$

It is clear that if $s = s_0$ (s_0 fixed mode) then $\frac{\partial \Phi}{\partial K_F} = 0.$

Finally with (3-4-5) we have $M(s_0,K_F) = 0$.

Proposition (3-4-1) shows that a multiple mode of order q may be a fixed mode of order $q^* \leq q$ if its order in the local subystems does not exceed $(q-q^*)$. Therefore, the simple modes of the overall system, which are also fixed modes, do not belong to the typical polynomials of closed loop local subsystems. Consequently, they belong to the set of modes of complementary subystems of the system. We may therefore reformulate proposition (3-4-1) according to the following corollary:

Corollary 3-4-3

If s_0 is a multiple mode of order q, then a necessary condition for s_0 to be a fixed mode relative to K_F is that the typical polynomial of the projection of the transfer matrix from the system to K_F^T has a zero in s_0 of a maximum order q-1.

Note that proposition (3-4-1) and corollary (3-4-3) are valid for any control structure, particularly for a centralized control. Corollary (3-4-3) expresses that an uncontrollable and/ora unobservable simple mode does not belong to the denominator of the transfer mode (minimal realization). Such a result is well known.

Example 3-4-2 /56

Let us take another look at example (3-4-1), the projection of the transfer matrix to K_F^T is:

$$\left[\begin{array}{c|c|c} 0 & \frac{1}{s-a} & 0 \\ \hline \frac{s-c}{s^2-cs-1} & 0 & 0 \\ \hline 0 & \frac{1}{(s-a)(s-b)} & \frac{1}{s-3} \end{array}\right]$$

it may therefore be stated according to corollary (3-4-3) that modes a,b, and 3 are not fixed.

If a decomposition is not imposed, the corollary (3-4-3) may select a decomposition without fixed modes.

Characterization of Distinct Modes

Given a system (2-4-4) with distinct modes, its transfer matrix may be decomposed as follows:

$$G(s) = \frac{A_1(s)}{s-s_1} + \sum_{i=2}^{n} \frac{A_i(s)}{s-s_i} \qquad A_1(s) \neq 0$$

$$= \frac{A_1(s)}{s-s_1} + \bar{G}(s) \qquad s_i \neq s_1 \quad i=2,\ldots,n \qquad (3\text{-}4\text{-}6)$$

Davison and Ozguner (DAV-83) (DAV-85b) gave the following theorem for systems with 2, 3 or 4 control stations. Similar results may be obtained for $N \geq 5$ using a recursive characterization (theorem 3-3-3).

Theorem 3-4-8 (DAV-83) (DAV-85b)

s_1 is a fixed mode of system (3-4-6) if and only if none of the following conditions is verifiable with respect to matrix A_1 and to matrix $\bar{G}(s_1) = \left[G(s) - \frac{A_1}{s-s_1}\right]_{s=s_1}$, or with respect to their transposition:

Case 1: (N=2)

$$\text{(i)} \quad A_1 = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \text{ and } \bar{G}(s_1) = \begin{bmatrix} X & X \\ 0 & X \end{bmatrix}$$

Case 2: (N=3) /57

$$\text{i)} \quad A_1 = \begin{bmatrix} 0 & 0 & X \\ 0 & 0 & X \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \bar{G}(s_1) = \begin{bmatrix} X & X & X \\ X & X & X \\ 0 & 0 & X \end{bmatrix}$$

$$\text{ii)} \quad A_1 = \begin{bmatrix} 0 & X & X \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \bar{G}(s_1) = \begin{bmatrix} X & X & X \\ 0 & X & X \\ 0 & X & X \end{bmatrix}$$

$$\text{iii)} \quad A_1 = \begin{bmatrix} 0 & X & 0 \\ 0 & 0 & 0 \\ 0 & X & 0 \end{bmatrix} \text{ and } \bar{G}(s_1) = \begin{bmatrix} X & X & X \\ 0 & X & 0 \\ X & X & X \end{bmatrix}$$

Case 3: (N=4)

$$\text{i)} \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & X \\ 0 & 0 & 0 & X \\ 0 & 0 & 0 & X \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \bar{G}(s_1) = \begin{bmatrix} X & X & X & X \\ X & X & X & X \\ X & X & X & X \\ 0 & 0 & 0 & X \end{bmatrix}$$

$$\text{ii)} \quad A_1 = \begin{bmatrix} 0 & 0 & X & X \\ 0 & 0 & X & X \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \bar{G}(s_1) = \begin{bmatrix} X & X & X & X \\ X & X & X & X \\ 0 & 0 & X & X \\ 0 & 0 & X & X \end{bmatrix}$$

$$\text{iii)} \quad A_1 = \begin{bmatrix} 0 & 0 & X & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & X & 0 \end{bmatrix} \text{ and } \bar{G}(s_1) = \begin{bmatrix} X & X & X & X \\ 0 & X & X & X \\ 0 & 0 & X & 0 \\ X & X & X & X \end{bmatrix}$$

iv) $A_1 = \begin{bmatrix} 0 & X & X & X \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\bar{G}(s_1) = \begin{bmatrix} X & X & X & X \\ 0 & X & X & X \\ 0 & X & X & X \\ 0 & X & X & X \end{bmatrix}$

v) $A_1 = \begin{bmatrix} 0 & X & X & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & X & X & 0 \end{bmatrix}$ and $\bar{G}(s_1) = \begin{bmatrix} X & X & X & X \\ 0 & X & X & 0 \\ 0 & X & X & 0 \\ X & X & X & X \end{bmatrix}$

vi) $A_1 = \begin{bmatrix} 0 & X & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & X & 0 & 0 \end{bmatrix}$ and $\bar{G}(s_1) = \begin{bmatrix} X & X & X & X \\ 0 & X & 0 & 0 \\ X & X & X & X \\ X & X & X & X \end{bmatrix}$

vii) $A_1 = \begin{bmatrix} 0 & X & 0 & X \\ 0 & 0 & 0 & 0 \\ 0 & X & 0 & X \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\bar{G}(s_1) = \begin{bmatrix} X & X & X & X \\ 0 & X & 0 & X \\ X & X & X & X \\ 0 & X & 0 & X \end{bmatrix}$

where X is an element that is not necessarily zero.

In the case of 2 control stations (N=2), the theorem may be interpreted as follows: after simplification, elements $G_{11}(s)$, $G_{21}(s)$ and $G_{22}(s)$ are not poles in s_1 and the elements of $G_{12}(s)$ have a zero in s_1, or, on the other hand, $G_{11}(s)$ and $G_{11}(s)$, $G_{12}(s)$ and $G_{22}(s)$ do not have a pole in s_1 and the elements of G_{21} (s) have a zero in Similar interpretations may be made for the other cases. /58

Example 3-4-3

Given example (3-4-1), with $a \neq b$, the transfer matrix may be decomposed as follows:

$$G(s) = \frac{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{b-a} & 0 \end{bmatrix}}{s-b} + \frac{\begin{bmatrix} 0 & (s^2-cs-1)(s-3) & 0 \\ (s-c)(s-a)(s-3) & 0 & 0 \\ 0 & \frac{(s^2-cs-1)(s-3)}{a-b} & (s-a)(s^2-cs-1) \end{bmatrix}}{(s^2-cs-1)(s-3)(s-a)}$$

For s=b we have:

$$A_1^T\ (s=b) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \bar{G}^T(s_1) = \left[G(s) - \frac{A_1}{s-b}\right]^T_{s=b} = \begin{bmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & 0 & * \end{bmatrix}$$

with*: nonzero parameter. A^T and $G^T(s_1)$ verify the condition i of case 2 of theorem (3-4-8), therefore s=b is a decentralized fixed mode of the system.

III.4.5. General Characterization in Terms of the Transfer Matrix

Anderson (AND-82) developed a complete characterization of decentralized fixed modes based on the results presented in section III.4.1. Given system (2-4-3), and subsets:

$$I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, p\} \text{ with } i_1 < i_2 < \cdots < i_k,$$
$$J = \{j_1, \ldots, j_k\} \subset \{1, \ldots, m\} \text{ with } j_1 < j_2 < \cdots < j_k$$

Let $\|I\|$ be the number of elements of set I, and $X\begin{bmatrix} I \\ J \end{bmatrix}$ be the minor of matrix X consisting of I lines and J columns.

Theorem 3-4-9 (AND-82)

Let us assume that by reordering the inputs and outputs, the transfer matrix of the system is expressed:

$$G(s) = \left[\begin{array}{c|c} G_{\alpha\alpha}(s) & G_{\alpha\beta}(s) \\ \hline G_{\beta\alpha}(s) & G_{\beta\beta}(s) \end{array}\right] = \left[S_\alpha(s)\quad S_\beta(s)\right]^{-1}\left[T_\alpha(s)\quad T_\beta(s)\right]$$

$$Y_i(s) = \sum_j G_{ij}(s)\, U_j(s) \qquad i,j = \alpha, \beta$$

If $P_\alpha(m_\alpha)$ ($P_\beta(m_\beta)$) are the number of lines (columns) of $G_{\alpha\alpha}$ ($G_{\beta\beta}$), then $P_\alpha = P_\alpha + P_\beta$ and $m = m_\alpha + m_\beta$.

Let us consider decentralized controls K = block diag. (K_α, K_β), and s_0 is assumed to be a multiple zero of order $\bar{q}$ of the typical polynomial of the system. Therefore the two following conditions are equivalent:

i) rank $[S_\alpha(s_o)\ T_\alpha(s_o)] < p_\alpha$

and the fixed mode s_0 is multiple of order q.

ii) s_0 is a multiple decentralized fixed mode of order q if 0 < q ≤ q exists so that when $n_r + n_c \geq p$, we have:

$$n_z \geqslant (q-\bar{q}) + (n_r + n_c - p_\alpha) \qquad (3\text{-}4\text{-}7)$$

for all G(s) minors. If $\bar{q} > q$, the equality is verified at least for one choice of I and J where $n_r + n_c = p_\alpha$.
with:

$n_z =$ | number of zeros in s_0 of minor $G\begin{bmatrix} I \\ J \end{bmatrix}$

$n_z < 0$ s_0 is a pole of order $- n_z$

$n_z = 0$ s_0 is neither a pole nor a zero

$n_z = \infty$ the minor is identically zero.

and $n_r = n_r\begin{bmatrix} I \\ J \end{bmatrix}$ number of lines among the p_α first lines which do not belong to the minor under consideration, and given by: $n_r\begin{bmatrix} I \\ J \end{bmatrix} = \| \bar{I} \cap \{1,\ldots,p_\alpha\} \|$ with $I \cup \bar{I} = \{1,\ldots,p\}$

and $n_c = n_c\begin{bmatrix} I \\ J \end{bmatrix}$ number of columns among the m_α first columns which belong to the minor under consideration, and given by:

$$n_c \begin{bmatrix} I \\ J \end{bmatrix} = \| J \cap \{1,\ldots, m_\alpha\} \|$$

The quantity $(n_r + n_c - P_\alpha)$ represents the position of the minor in the transfer matrix.

According to theorem (3-4-9) there is a multiple fixed mode s_0 of order q if and only if certain minors have a zero in s_0 of a minimum order or a limited multiplicity pole, and if s_0 is at the same time a G(s) pole.

Anderson's theorem is complicated in its general form, but it may used in a simple form for special cases, particularly for systems with simple modes as shown in the following theorem:

Theorem 3-4-10 (AND-82) (OZG-83) /60

If s_0 is a simple mode of the system, then with the same assumptions as in theorem (3-4-9), s_0 is a decentralized fixed mode of the system if and only if the transfer matrix of the system (or its transposition) may be expressed:

$$G(s) = \left[\begin{array}{c|c} \text{no element has a pole in } s_0 & s_0 \text{ is a simple zero of the typical polynomial of this block.} \\ \hline \text{each element has a zero in } s_0 & \text{no element has a pole in } s_0 \end{array}\right]$$

Example 3-4-4

Given example (3-4-1) with a=c and b=3; the transfer matrix is:

$$\left[\begin{array}{c} y_1 \\ \hline y_2 \\ \hline y_3 \end{array}\right] = \left[\begin{array}{c|c|c} 0 & \frac{1}{s-a} & 0 \\ \hline \frac{s-a}{s^2-as-1} & 0 & 0 \\ \hline 0 & \frac{1}{(s-3)(s-a)} & \frac{1}{s-3} \end{array}\right] \left[\begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array}\right]$$

the typical polynomial is $\Phi(s) = (s-E)^2\ (s^2-as-1)$

Simple Mode: $s = a$

By reordering the inputs and outputs, matrix $G(s)$ is expressed:

$$\begin{bmatrix} y_1 \\ y_3 \\ \hline y_2 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 0 & \frac{1}{s-a} \\ & \frac{1}{s-3} & \frac{1}{(s-3)(s-a)} \\ \hline \frac{s-a}{s^2-as-1} & 0 & 0 \end{array}\right] \begin{bmatrix} U_1 \\ U_3 \\ \hline U_2 \end{bmatrix}$$

and by combining subsystems 1 and 3, theorem (3-4-10) gives a fixed mode in a.

Multiple Mode: $s = 3$

By reordering the inputs and outputs, the transfer matrix is expressed:

$$\begin{bmatrix} y_3 \\ \hline y_1 \\ y_2 \end{bmatrix} = \left[\begin{array}{c|cc} \frac{1}{s=3} & 0 & \frac{1}{(s-3)(s-a)} \\ \hline 0 & 0 & \frac{1}{s-a} \\ 0 & \frac{s-a}{s^2-as-1} & 0 \end{array}\right] \begin{bmatrix} U_3 \\ \hline U_1 \\ U_2 \end{bmatrix}$$

and by combining subsystems 1 and 2, we have: /61

$$p_\alpha = 1, \quad p_\beta = 2 \text{ and } P = P_\alpha + p_\beta = 3$$
$$m_\alpha = 1, \quad m_\beta = 2 \text{ and } m = m_\alpha + m_\beta = 3$$

and the mode under consideration is multiple of order $\overline{q} = 2$.

The minors of $G(s)$ verifying $n_r + n_c \geq p_\alpha$ are: $G\begin{bmatrix}1\\1\end{bmatrix}$, $G\begin{bmatrix}2\\1\end{bmatrix}$, $G\begin{bmatrix}2\\2\end{bmatrix}$, $G\begin{bmatrix}2\\3\end{bmatrix}$, $G\begin{bmatrix}3\\1\end{bmatrix}$, $G\begin{bmatrix}3\\2\end{bmatrix}$, $G\begin{bmatrix}3\\3\end{bmatrix}$, $G\begin{bmatrix}12\\12\end{bmatrix}$, $G\begin{bmatrix}12\\13\end{bmatrix}$, $G\begin{bmatrix}13\\12\end{bmatrix}$, $G\begin{bmatrix}13\\13\end{bmatrix}$, $G\begin{bmatrix}23\\12\end{bmatrix}$, $G\begin{bmatrix}23\\13\end{bmatrix}$, $G\begin{bmatrix}23\\23\end{bmatrix}$ & $G\begin{bmatrix}123\\123\end{bmatrix}$.

According to theorem (3-3-9) $s=3$ is a fixed mode of order q if and only if a q exists so that (without considering the minors including $z = \infty$ since condition 3-4-7 is then always verified):

$$z = 1 \geq q - 2$$
$$z = 0 \geq q - 2$$

the two inequalities are verified for q=1 (not for q=2) and therefore s=3 is a simple fixed mode for the division under consideration.

C - The case of a decentralized control with three stations is included in cases A and B, therefore the system has two fixed modes in a and b=3.

III.4.6. Interpretation

The characterizations of fixed modes in the frequency range area divided into two groups:

1) According to the methods the fixed modes can be characterized by transmission zeros of some subsystem s(see III.4.2). These methods are a reformulation of the characterizations of section III.3.3 in the frequency range: a fixed mode is a transmission zero which corresponds to the frequency which cuts the information flow between the inputs and outputs of certain complementary subsystems (zero transfer matrix $G(s_o) = 0$.

2) Characterizations making it possible to interpret the appearance of fixed modes in terms of uncontrollable and/or unobservable modes (see III.4.1), III.4.4., III.4.5).

Let us consider the special case of a system with two single input/single output systems described by:

$$G(s) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \tag{3-4-8}$$

By applying a control $u_2 - k_2 y_2$ to the 2nd subsystem, the transfer function seen from the first station is: /62

$$\frac{y_1(s)}{u_1(s)} = g_{11}(s) + g_{12}(s)\,\frac{k_2}{1-g_{22}(s)\,k_2}\,g_{21}(s) \qquad (3\text{-}4\text{-}9)$$

Figure (3.2.a) illustrates this relationship.

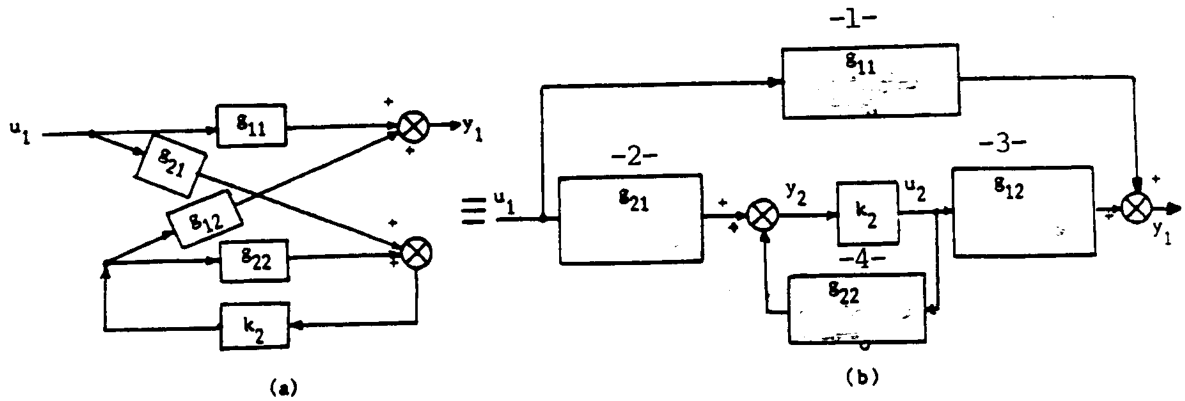


Fig. 3.2. - Illustration of a Fixed Mode Giving an Uncontrollability

Key: 1-no pole in s_0; 2-zero in s_0; 3-pole in s_0;
4-no pole in s_0.

For this special case, Anderson's theorem (3-4-10) may take on the following form:

Theorem 3-4-11 (AND-82)

Given system (3-4-8), then s_0 is a simple decentralized fixed mode of the system if and only if the transfer matrix of the system or its transposition has the following form:

$$G(s) = \left[\begin{array}{c|c} \text{no pole in } s_0 & \text{pole in } s_0 \\ \hline \text{zero in } z_0 & \text{no pole in } s_0 \end{array}\right]$$

Figure (3.2b) illustrates the theorem and shows that to have a zero in s_0 for g_{21} and a pole in s_0 for g_{12} reveals an uncontrollability (see II.2.4). The appearance of a decentralized fixed mode may be interpreted as a zero-pole (or pole-zero) interpretation in the product g_{21} $g_{21} \cdot \frac{k_2}{1-g_{22}k_2} \cdot g_{12}$, therefore by a decentralized uncontrollability (unobservability). For the case under consideration (simple mode) this uncontrollability (unobservability) should not be provided by direct chains (g_{11} and g_{22} do not have a pole in s_0). This result corresponds exactly with Davison and Ozguners' theorem (3-4-8) and with the condition of proposition (3-4-1). Theorem (3-4-11) shows that if G(s) is natural and s_0 is a fixed mode, then the system should have at least three poles, two among those may coincide, but not in s_0.

If s_0 is a multiple fixed mode of the system, then the direct chains (g_{11} and/or g_{22}) may have a pole in s_0 of an /63
order lower than the order s_0 in one of the typical polynomials of the transfer functions between stations (g_{21} and/or a g_{12}) (AND-82). In this case, the pole-zero simplification is done between a g_{22} pole, or a block zero $\frac{k_2}{1-g_{22}k_2}$, , and a g_{12} and/or g_{21} pole, which results in a decentralized uncontrollability or unobservability

In the general case of simple fixed modes theorem (3-4-10) reveals a partition into two aggregates stations α and β, a partition already found in the characterization in the time range. The interpretation can be made in a similar manner to the case of the single input/single output subsystems: the pole-zero simplification is done between a pole of block $G_{\alpha\beta}$ (or $G_{\beta\alpha}$) and a zero common to all elements of block $G_{\beta\alpha}$ (or $G_{\alpha\beta}$).

Figure (3.2b) also shows that the fixed mode remains so even if k_2 is replaced by a transfer function because this change does not prevent the poles and zeros of g_{21} and g_{12} from being simplified. Such a result was already; found by Wang and Davison (WAN-73b) and Corfmat and Morse (COR-76). Note that if

k_2 is changed into an unsteady control, it will no longer be possible to simplify: this idea is used to eliminate the fixed modes (see chapter VI).

Finally, note that if $g_{12} = 0$ (or $g_{21} = 0$) the modes g_{21} (g_{12}) will be uncontrollable for the looped system (3-4-9) (therefore they remain fixed). Moreover, their existence is independent of the values of the system parameters. These modes are called **STRUCTURAL FIXED MODES**, details of their study are given in the next chapter.

III.5. CONCLUSION

We have described in this chapter the algebraic methods for characterizing the fixed modes in the time and frequency ranges, and have shown the links between these different characterizations. Thanks to these methods, we were able to interpret the fixed modes either in terms of system transmission zeros or in terms of controllability and observability.

It is evident that the methods presented here are of unequal importance, but that each contributed to giving part of the present results. Let us stress for example that Seraji's characterization (SER-82) is impractical for general cases, but that he was the first to develop typical polynomials of the closed loop system to characterize the fixed modes in terms of system zero. This same development was used later by Vidysagar and Viswanadham (VID-82,83), Tarokh (TAR-84) and Hu-Jiang (HUJ-84), either to characterize the fixed modes in a more direct way in terms of transmission zeros (TAR-84), or to provide a more systematic method of calculating fixed modes in the frequency range (VID-82), 83). Note that Davison and WAng (DAV-85a) recently gave a new more direct characterization in terms of transmission zeros.

Studies by Davison et al (DAV-83, see III.3.2 and III.4.4) are of limited interest compared to other methods. Conversely, those of Anderson et al (AND-81a, 82) are important because they show that the control and observation stations are partitioned (for a system with a fixed mode) into control and observation stations for which the fixed mode is simultaneously uncontrollable by one and unobservable by the other. They therefore show that the notion of fixed modes is an extension of the notion of controllability and observability.

Finally, we personally proposed a method of calculating fixed modes in the time range based on the notion of the sensitivity of natural values in a closed loop (see III.2.2), and in the frequency range, a necessary condition for the existence of fixed modes (see III.4.3.) permitting a decomposition without fixed modes.

CHAPTER IV - STRUCTURAL FIXED MODES /65

IV.1 INTRODUCTION

Knowing that the values of the parameters associated with a physical system are never accurately known, Sezer and Siljak (SEZ-81a) (SEZ-81b) demonstrated that the fixed modes may come from two sources: equality between certain parameters of the system (in this case a small parameter disturbance is enough to eliminate them) or the structure itself of the system. In this case, the fixed modes are called **STRUCTURAL** ones. They can only be removed by changing the structure of the system. The following example shows the difference between these two types of modes.

Example 4-1-1 (SEZ-81a,b) /66

Given example (3-3-2) and a decentralized control K = diag(k_1,k_2), the closed loop dynamic matrix is:

$$D = A+BKC = \begin{bmatrix} 0 & 1 & k_1 \\ 1 & 1 & 0 \\ k_2 & 0 & 1 \end{bmatrix}$$

Expressing the typical equation:

$$(\det (sI-D) = (s-1)\ (s^2-2-1-k_1k_2)$$

we see that the system has a natural value s=1 which is independent of the k_1 and k_2: s=1 is a decentralized fixed mode of the system.

If ϵ is a small variant in an element of A which becomes:

$$A_\varepsilon = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1+\varepsilon \end{bmatrix}$$

then: $\det(sI-D_\varepsilon) = s(s-1)\ (s-1-\varepsilon) - (s-1-\varepsilon) - k_1k_2(s-1)$

and the fixed mode s=1 disappears; it was nonstructural. Conversely if A:

$$A = \begin{bmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then $\det\ (sI-D) = s\ (s^2-ab-k_1k_2)$

and we have a fixed modes=0, which will remain for any variation in the nonzero elements of matrices A, B and C. This is a structural fixed mode.

Sezer and Siljak (SEZ-81a and b) then divide the fixed modes into two types; **STRUCTURAL FIXED MODES** (MFS) and **NONSTRUCTURAL FIXED MODES** (MFNS). By this distinction they generalize the notion of fixed modes in that the structural fixed modes (*) are modes of the system that remain invariant under any nonzero parametric variation of the system, not just the control parameters. Note that this notion is derived by generalizing the notion of structural controllability under structural stress, that we are going to study in the next section:

(*) See def. 4-3-3 for a more accurate definition.

IV.2 STRUCTURAL CONTROLLABILITY AND OBSERVABILITY

The notion of structural controllability was introduced by Lin (LIN-74) in 1974 for single input nonlinear systems. Shield and Person in 1976 (SHI-76) applied his results to multi-input linear systems using algebraic methods. In the same year, Glover and Silverman (GLO-76) considerably simplified Shield and Pearsons' conditions by reducing their algebraic results to simple operations on Boolean matrices. Davison (DAV-77) was interested in structural observability as a dual notion of structural controllability.

In the structural approach, only the zero elements of the system matrices due to the absence of physical connections between certain parts of the system are considered to be perfectly determined, and therefore fixed. The other elements are considered to indicate only that connections exist.

M is said to be a structured matrix if its elements are either fixed zeros, or independent parameters (SHI-76).

Definition 4-2-1 (SHI-76) **Structurally Equivalent Matrix**

Two matrices are said to be structurally equivalent if and only if the positions of the zero elements of one correspond exactly with the positions of the zero elements of the other.

Definition 4-2-2 (SHI-76) **Structural Rank**

The structural rank of a real matrix M, notated gr(M)*, is the maximum rank that a matrix equivalent to M can have. It is determined only as a function of its zero elements, independently of the values taken on by its nonzero elements (see Appendix 2).

(*) gr: as in "generic rank" in English.

Definition 4-2-3 (SHI-76) (DAV-77)

The system (C,A,B) is structurally controllable and observable if there is a controllable and observable system $(\bar{C},\bar{A},\bar{B})$ structurally equivalent to (C,A,B).

Theorem 4-2-1 (SHI-76), (CHO-82)

The system (C,A,B) is structurally controllable and observable if and only if none of the two following conditions is verified:

i) a permutation matrix P exists so that C,A and B may be put in the form:

ii)

$$PAP^T = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \qquad PB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

$$CP^T = \begin{bmatrix} C_1 & 0 \end{bmatrix}$$

$$gr \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} < n$$

Davison (DAV-77), Franksen et al (FRA-79a and b) and /68

Reinschke (REI-81) reformulated this theorem in terms of graph theory. This formulation is given in chapter VII (theorem 7-3-1).

Finally, note that Mortazavian (MOR-82) introduced the notion of structural (observabililty) controllability index as a concept equivalent to the standard (observability) controllability index. It showed that the standard controllability index is greater than or equal to the structural controllability index. For more details the reader is referred to (MOR-82).

IV.3 DECENTRALIZED CONTROLLABILITY AND OBSERVABILITY

Kobayashi et al (KOB-78) and Momen and Evans (MOM-83) defined decentralized controllability as follows:

Definition 4-3-1) (KOB-78) (MOM-83) **Decentralized Controllability**

A system with N control stations is controllable under decentralization stresses if a decentralized output feedback control exists that transfers any initial state of the system to the origin in a finite time interval T, i.e. $X(T) = 0 \; \forall \; X(o)$.

By defining the states that are controllable and observable by station i as the controllability subspace of station i, and the states that are controllable and observable by station i in conjunction with other stations (by looping the stations, with the exception of i) as the expanded controllability subspace of station, the following theorem is proposed.

Theorem 4-3-1 (KOB-78)

The system (2-4-1) with N control stations is controllable in a decentralized manner if and only if the union of the expanded controllability subspaces of all stations form the state space of the system.

Definition 4-3-2 (MOM-83) Decentralized Structural Controllability

System (2-4-1) is structurally controllable under decentralization stresses if and only if a system exists that is structurally equivalent to (2-4-1), and controllable in a decentralized manner.

Momen and Evans (MOM-83) using the notions of graph theory give a structural version of theorem (4-3-1).

Results by Kobayashi and Yoshikawas (KOB-82), Momen and Evans (MOM-83) are very interesting, because they directly associate decentralized uncontrollability with the existence of fixed modes. For a system with N control stations, Kobayashi and Yoshikawa KOB-82) define the quotient system as a station regrouping into q classes ($q \le N$) where each class is made up of stations that have nonzero transfer matrices between them. (In terms of graph theory, a class is made up of highly connected stations). The transfer matrix of the quotient system is therefore block-triangular. Kobayashi and Yoshikawa propose the following theory:

Theorem 4-3-2 (KOB-82)

The system (2-4-1) is structurally controllable under decentralized structural stresses if and only if the quotient system does not have fixed modes.

Definition 4-3-3 (SEZ-81) **Structural Fixed Modes**

System (2-4-1) has structural fixed modes with respect to control K_F if and only if all systems structurally equivalent to (2-4-1) have fixed modes with respect to the same control.

Momen and Evans (MOM-83) give the following theorem:

Theorem 4-3-3 (MOM-83)

System (2-4-1) has no structural modes with respect to the decentralized control if and only if it is structurally controllable and structurally observable under the decentralization stress.

The first interesting conclusion is that the structural fixed modes of a system are fixed modes of the quotient system: they are structurally uncontrollable under decentralization stresses. This uncontrollability is due to the lack of information transfer between the subsystems (zero blocks in the transfer matrix). The second conclusion is that a highly connected system (def. 2-4-4) is always structurally controllable in a decentralized manner and therefore never has structural fixed modes. If it has fixed modes then they are nonstructural. This conclusion is confirmed by Chong (CHO-82).

Chong (CHO-82) borrows Corfmat's and Morse's method (COR-76a,b). (See II.4.3b) and gives the equivalent of their results from the structural standpoint.

Theorem 4-3a-4 (CHO-82)

Since system (2-4-1) is controllable and observable, a decentralized control exists making the system structurally controllable and observable by a single station if and only if all complementary subsystem of the system are structurally complete: /70

i) $$\text{gr}\begin{bmatrix} A & B_\alpha \\ C_\beta & 0 \end{bmatrix} \geqslant n$$

ii) the system is highly connected, i.e. $C_\beta(sI-A)^{-1} B_\alpha \neq 0\ \forall_\alpha$. or α and β are defined in def. (2-4-3).

IV.4 ALGEBRAIC CHARACTERIZATION OF STRUCTURAL FIXED MODES

Using the results developed for structural controllability and observability (LIN-74) (SHI-76) (GLO-76) Sezer and Siljak (SEZ-81a) characterized structural fixed modes by the following theorem.

Theorem 4-4-1 (SEZ-81a)

System (2-4-1) has structural fixed modes with respect to a decentralized control Kd if and only if one of the two following conditions is proved:

i) $\alpha \subset N$ and a permutation matrix P exist such that:

$$P^T A P = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad P^T B_\alpha = \begin{bmatrix} B_\alpha^1 \\ B_\alpha^2 \\ B_\alpha^3 \end{bmatrix}, \quad P^T B_\beta = \begin{bmatrix} 0 \\ 0 \\ B_\beta^3 \end{bmatrix}$$

$$C_\alpha P = \begin{bmatrix} C_\alpha^1 & 0 & 0 \end{bmatrix}$$
$$C_\beta P = \begin{bmatrix} C_\beta^1 & C_\beta^2 & C_\beta^3 \end{bmatrix} \tag{4-4-1}$$

ii) $\alpha \subset \mathbf{N}$ exists such that:

$$gr = \begin{bmatrix} A & B_\alpha \\ C_\beta & 0 \end{bmatrix} < n$$

where sets α and β are two natural subsets of the set

$$\mathbf{N} = \{1,2,\ldots,N\}, \text{ with } \mathbf{N} = \alpha \cup \beta$$

Note that this theorem may be generalized for a control under any structural stress. Simply substitute α and β by I and J (see def. 2-4-7) (PIC-84).

Theorem (4-4-1) establishes two types of structural fixed modes: type i which corresponds to the modes of block A_{22} in (4-4-1) and type ii which corresponds to the fixed modes at the origin. These are independent of the values of the system parameters, not just of the control.

The diagram of figure 4.1 illustrates condition i of theorem 4-4-1 and shows that for a system with type i fixed modes, the state space is divided into three components X_1, X_2 and X_3. States X_2 correspond to the fixed modes and are, for a partition of control and observation stations into two aggregated stations, α & β, uncontrollable by station β and unobservable by α. The block-triangular structure of the system matrices makes the transfer matrix of the system (after reordering the inputs and outputs if necessary) is block-triangular, and therefore the information transfer from station β to station α is not provided (see fig. 4.1). In this case, the fixed modes are structurally

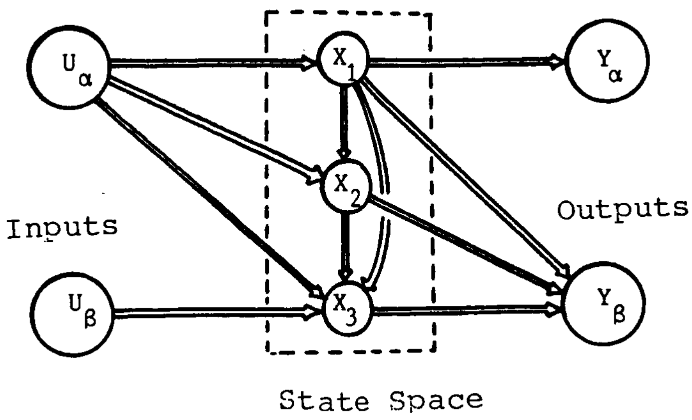


Fig. 4.1 - Accessibility in a system
With Structural Fixed Modes

uncontrollable under decentralization stresses.

Type ii structural fixed modes are modes at the origin. The transfer matrix of a system with a simple fixed mode may appear in two forms (see theorem 3-4-10):

Either it is block-triangular:

$$G(s) = \left[\begin{array}{c|c} G_{\alpha\alpha} & 0 \\ \hline G_{\beta\alpha} & G_{\beta\beta} \end{array}\right]$$

where $G_{\beta\alpha}$ has a mode at the origin. This mode is of the same nature as type i structural fixed modes. It belongs to the modes of the quotient system and is therefor structurally uncontrollable under structural stress. These modes are therefore called type iil structural fixed modes.

Or it is highly connected, i.e.:

$$G(s) = \left[\begin{array}{c|c} G_{\alpha\alpha} & G_{\alpha\beta} \\ \hline G_{\beta\alpha} & G_{\beta\beta} \end{array}\right]$$

where $G_{\beta\alpha}$ has a mode at the origin and all elements of $G_{\alpha\beta}$ have a zero at the origin. In this case the fixed mode can be of two different types: if the zero at the origin of one of the elements is the result of a simplification of type as+b-c with b=c, the mode is then nonstructural (see example 4-4-1) (a small disturbance of parameters b or c is enough to eliminate it). /72
However if zero is not the result of such a simplification, then the mode is structural because its position at the origin makes it independent of the system parameters. However, ita is comparable to nonstructural fixed modes because it is the result of a pole-zero (zero-pole) simplification. These modes are therefore called type ii_2 structural fixed modes (example 4-4-2). Note that owing to the highly related property of the transfer matrix (def. 2-2-4) these fixed modes are structurally controllable under decentralization stresses. Therefore an unsteady control can eliminate them (KOB-78) (AND-81b) (see ch. VI).

Example 4-4-1)

Let us consider a system with two control stations expressed by:

$$\dot{X} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 2 \end{bmatrix} X + \left[\begin{array}{c|c} 1 & 0 \\ 0 & -1 \\ 0 & 2 \end{array}\right] \begin{bmatrix} u_1 \\ \hline u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ \hline y_2 \end{bmatrix} = \left[\begin{array}{ccc} 0 & 1 & 1 \\ \hline 1 & 0 & 0 \end{array}\right] X$$

The transfer matrix of the system is:

$$G(s) = \left[\begin{array}{c|c} 0 & \dfrac{s+2(1-a_{22})}{(s-a_{22})(s-2)} \\ \hline \dfrac{1}{s} & 0 \end{array}\right]$$

it is clear that the system has a fixed mode at the origin only if $a_{22} = 1$. Therefore s=0 is a nonstructural fixed mode.

Example 4-4-2

Let us consider a system with two control stations expressed by:

$$\overset{o}{X} = \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X + \left[\begin{array}{c|c} b_{11} & 0 \\ 0 & 0 \\ 0 & b_{32} \end{array}\right] U$$

$$Y = \left[\begin{array}{ccc} 0 & 0 & c_{13} \\ \hline c_{21} & 0 & 0 \end{array}\right] X$$

The transfer matrix of the system is:

$$G(s) = \left[\begin{array}{c|c} 0 & \dfrac{c_{13}b_{32}}{s} \\ \hline \dfrac{b_{11}c_{21}s}{s^2 - a_{21}a_{12}} & 0 \end{array}\right]$$

The system has a fixed mode at the origin for any system /73
parameter variation is highly connected, therefore the fixed mode is structural of type ii_2.

IV.5 DETECTION OF STRUCTURAL FIXED MODES

In this section, we shall apply the results of section III.2.2 to calculate the sensitivity of structural fixed modes. Let us reconsider the gradient of a distinct natural value with respect to variations of the matrix elements D (equation 3-2-7).

$$\frac{d\, s_r}{d(d_{ij})} = W_r^T \frac{dD}{d(d_{ij})} V_r \tag{3-2-7}$$

where W_r and V_r are left and right natural vectors of matrix D and correspond to s_r. Let us assume that elements d_{ij} and D depend on physical parameters R_1, $R_2, \ldots,$ R_Q. If R is one of these parameters then the derivative of D with respect to R is expressed:

C-2

$$\left(\frac{\partial D}{\partial R}\right)_{ij} = \begin{cases} \rho_{ij}(R)\ I^i\ I_j \\ 0 \quad \text{if} \quad d_{ij} = 0 \end{cases}$$

with $\rho_{ij}(R) = \dfrac{\partial d_{ij}(R)}{\partial R}$ (4-5-1)

where I^i and I^j are respectively the i-th column and j-th column of the unity matrix. By carrying (4-5-1) to (3-2-7) we obtain:

$$\left(\frac{\partial s_r}{\partial R}\right)_{ij} = \begin{cases} \rho_{ij}(R)\ W_r\ I^i\ I_j\ V_r \\ 0 \quad \text{if} \quad d_{ij} = 0 \end{cases} = \begin{cases} \rho_{ij}(R)\ w_i\ v_j \\ 0 \quad \text{if} \quad d_{ij} = 0 \end{cases}$$

where w_i and v_j are the i-th and j-th components of the left and right vectors corresponding to s_r respectively. By generalizing this relation, we obtain the gradients of a natural value with respect to the parameters $R_1, \ldots, R_q$:

$$\left(\frac{\partial s_r}{\partial R_t}\right)_{ij} = \begin{cases} \rho_{ij}(R_t)\ w_i\ v_j & i,j=1,\ldots,n \\ 0 \quad \text{if} \quad d_{ij} = 0 & t=1,\ldots,q \end{cases} \qquad (4\text{-}5\text{-}2)$$

In a general case, the derivatives $\rho_{ij}(R_t)$ are not all zero at the same time, we therefore find that a natural value is insensitive to the variations of R_t $t=1,\ldots,q$ if and only if $w_i.v_j = 0$. This makes it possible to state the following /74 lemma:

Lemma 4-5-1

A necessary and sufficient condition for a simple natural value s_r of a real matrix to be insensitive to the variations of the elements of matrix D is: $w_i.v_j = 0$, $i,j=1,\ldots,n$ where w_i and v_j are the i-th and j-th components of the left and right natural D vectors corresponding to s_r.

Definition 4-5-1 (TAR-84a) Structural Sensitivity Matrix

The structural sensitivity matrix SS_r of a simple natural value s_r is defined by:

$$SS_r = \| ss_{ij} \|_{\, i,j=1,..,n}$$

with

$$ss_{ij} = \begin{cases} 1 \text{ if } w_i \cdot v_j \neq 0 \\ 0 \text{ if } d_{ij} = 0 \\ 0 \end{cases}$$

The following proposition characterizes structural fixed modes:

Proposition 4-5-1 (TAR-84a)

Given a closed loop system (3-1-2) with distinct modes, then s_r is a structural fixed mode with respect to control K_F if and only if one of the following conditions is proved:

1) The structural sensitivity matrix SS_r corresponding to s_r is identical to one of the structural sensitivity matrices of the set of matrices of systems structurally equivalent to system (3-1-2).

2) The structural sensitivity matrix SS_r corresponding to s_r is identically zero.

Demonstration

It is clear that the two conditions of this proposition are equivalent to the conditions of Sezer and Siljaks' theorem (4-4-1). Condition 1 corresponds to type i fixed modes that are modes of block A_{22} (equation 4-4-1). These modes are also sensitive to variations of the elements of this block. Consequently, the structural sensitivity matrices are equivalent for systems which have type i structural fixed modes. For condition 2, the modes that prove it are modes at the origin and they are independent of any variation of the system elements. According to Lemma 4-5-1, it may be concluded that the structural sensitivity matrix corresponding to $s_r = 0$ is identically zero. /75

The following algorithm provides a numerical method for detecting fixed modes and defining their nature.

Algorithm 4-5-1

To find the fixed modes of a system with distinct modes, relative to the set of K_F controls and to determine their origin:

1 - Select a control matrix $K \in K_F$ so that the modes of a closed loop system $\sigma(D) = \sigma(Aa+BKC)$ are distinct.

2 - for any $s_r \in \sigma(D)$, calculate the sensitivity matrices relative to control SK_r (defined in 3-2-10). If SK_r is zero, then the natural corresponding value is a fixed mode, i.e. $s_r \in \Lambda$

3 - If Λ is an empty set go to 12.

4 - For any $s_r \in \Lambda$, calculate the structural sensitivity matrix SS_r; if SS_r is zero, then the corresponding natural value s_r is a type ii structural fixed mode, i.e. $s_r \in \Lambda_{S2}$.

5 - Do $\Lambda_1 = \Lambda - \Lambda_{S2}$.
If Λ_1 is an empty set, go to 11.

6 - Select a matrix $\bar{D}$ that is structurally equivalent to D so that the natural values $\sigma(\bar{D})$ are distinct.

7 - Determine the fix modes $\bar{D}$ by returning to phase 2, or $\bar{\Lambda}$.
If $\bar{\Lambda}$ is an empty set do $\Lambda_{NS} = \Lambda_1$ and go to 11.

8 - For any $s_r \in \Lambda$ calculate the structural sensitivity

matrices $\overline{SS}_r$; if $\overline{SS}_r$ is zero for certain s_r, then these are type ii structural fixed modes of system D, i.e. $s_r \in \bar{\Lambda}_{S2}$.

9 - Do $\bar{\Lambda}_1 = \bar{\Lambda} - \bar{\Lambda}_{S2}$.

If $\bar{\Lambda}_1$ is empty then $\Lambda_{NS} = \Lambda_1$ and go to 11.

10 - For any $s_r \in \Lambda_1$, compare SS_r with the set of matrices $\overline{SS}$; if a matrix SS exists that is equivalent to SS_r, then s_r is a type i structural fixed mode, i.e. $s_r \in \Lambda_{S1}$.

11 - -The fixed modes of the system are:

Λ : set of fixed modes.

Λ_{S1} : set of type i structural fixed modes.

Λ_{S2} : set of type ii structural fixed modes.

Λ_{NS} : set of nonstructural fixed modes.

12 - Stop

Remark: A corresponding FORTRAN program is given in (TIT-86).

Example 4-5-1 /76

Given example (3-3-1) with q = c = 2 and b = 1, and K = diag (1,2,1); we have:

$$D = A+BKC = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

by applying algorithm (4-5-1) we get the results of tabl. (4.1)

1*	4	1	-1	3	2
SK	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}-0.4835&0&0\\0&-0.2417&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0.4905&0&0\\0&0.2417&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0.5275\ 10^{-15}&0&0\\0&-0.3925\ 10^{-16}&0\\0&0&0\end{bmatrix}$
Λ		M.F.			M.F.
SS		$\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$			$\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$

*B.F. Modes

Table 4.1

The fixed modes of the system are The equivalent system selected is:

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0,25 \\ 0 & 1.4 & 0.4 & 0 & 0 \\ 0 & 0 & 0.65 & 0.7 & 0 \\ 1.6 & 0 & 0 & 1.9 & 0 \\ 1.05 & 0 & 0 & 0 & 2.5 \end{bmatrix}$$

Table (4.2) gives the results on the equivalent system.

By comparing matrices SS and SS we have: $\Lambda_{S1} = \{1\}$
type i structural fixed modes $\Lambda_{NS} = \{2\}$
and nonstructural fixed modes

IV.6 COMMENTS

Sezer's and Siljak's characterizations (SEZ-81b) may be used to interpret the structural fixed modes, by generalizing Anderson's and Clement's results (AND-81a): A structural fixed mode (type i and ii_1) is, by regrouping the stations into two aggregated stations, structurally uncontrollable by one of the two

*	1.4	0.65	0.24483	-2.620605	2.024226
SK	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}-0.5669&0&0\\0&-0.07086&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0.1105&0&0\\0&0.0149&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}-0.6333&0&0\\0&-0.07916&0\\0&0&0\end{bmatrix}$
$\bar{\lambda}$		M.F.			
SS			$\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$		

*B.F. Modes

Table 4.2

stations and that are stucturally unobservable by the other. These modes can only be suppressed by changing the control structure (see ch. VII). Conversely, a type ii2 structural fixed mode is structurally controllable under structural stresses and an unsteady control can suppress it (see ch. VI).

The drawback of Sezer's and Silajk's characterization (from the standpoint of a numerical test for the existence of such a mode) is the necessity of finding a permutation matrix P, which is not the case for a characterization by sensitivity analysis. Pichai et al (PIC-83b) provided an algorithm for hierarchically breaking down a system into subsystems accessible by inputs and outputs, based on graph theory notions, and that can be used in searching for matrix P. This algorithm does not give just one solution and and is not valid in all cases. Note that by using the structural sensitivity matrix, the state set X2 (see eq. 4-4-1) can be easily found by using the structural sensitivity matrix, and the partition of the control and observation stations can therefore be determined, into two aggregated stations. It is therefore possible to also determine the set of sufficient matrix elements K to suppress type i structural fixed modes (see ch.VII).

CHAPTER V - GRAPHIC CHARACTERIZATION OF FIXED MODES

I. INTRODUCTION

In this chapter, we shall present the currently existing graphic methods for characterizing fixed modes. The use of graphs gives certain advantages, among which:

.Use of qualitative properties of the system,
.Binary type numerical calculations more adapted to a computer,
.Use of already existing results and algorithms of the graph theory,
.The use of graphs makes it easier to consider general structural stresses (see II.4.5).

The graphs used are directed ("Directed graph = digraph") and /80
have the following elementary components: peaks, arcs, loops, paths and circuits.

Definition 5-1-1 Graph Components

PEAK	:The points on a graph are called peaks
ARC	:A line joining two peaks
LOOP	:An arc whose origin and end are indistinguishable
PATH	:A series of adjacent arcs used for passing from one peak to another
SOURCE	:The initial peak of a path is called source.
WELL	:The final peak of a path is called well.
CIRCUIT	:A path whose source coincides with the well (closed path). A path or circuit is called:
ELEMENTARY	:if it passes only one time by each of its peaks
SIMPLE	:if it passes only one time by each of its arcs.

Many algorithms are found in literature for determining the paths or circuits of a graph, among which: (KRO-67), (KAU-68), (SRI-79) and (KAR-84).

CHARACTERIZATION USING ELEMENTARY CIRCUITS

Let's look at system (2-4-4) which has distinct modes and a transfer matrix $G(s) \in R^{p \times m}$. Let F be the matrix representing the structure of the acceptable control, i.e. $(j,i) \notin F$ if $k_{ij} = 0$. The directed graph $D_F = (V_F, E_F)$ is associated with the system and with F and its peaks V_F and arcs E_F are partitioned as follows:

$$V_F \triangleq V_{1F} \cup V_{2F} \quad \text{and} \quad E_F \triangleq E_{1F} \cup E_{2F}$$

with

$$\begin{aligned} V_{1F} &\triangleq \{i : (j,i) \in F \text{ for a certain } i\} \\ V_{2F} &\triangleq \{j : (j-m,i) \in F \text{ for a certain } j\} \\ E_{1F} &\triangleq \{(i,j) : (i,j) \in V_{1F} \times V_{2F},\ G_{j-m,i}(s) \neq 0\} \\ E_{2F} &\triangleq \{(i,j) : (i-m,j) \in F\} \end{aligned}$$

where m is the number of inputs of the system and $G_{j-m,i}$ is the transfer function between the i-th input and the (j-m)-th output. Each graph peak represents an input or an output, and each arc represents a nonzero transfer function or an enabled output-input connection (enabled feedback). A transmittance $t_{i,j}(s)$ is associated with each arc $(i,j) \in E_F$ according to the rule:

$$t_{i,j}(s) = \begin{cases} G_{j-m,i}(s) & \text{if } (i,j) \in E_{1F} \\ 1 & \text{if } (i,j) \in E_{2F} \end{cases}$$ /81

and the transmittance of a path T is the transmission product $t_{i,j}$ of the arcs forming T. With these definitions Locatelli et al (LOC-77) characterize the fixed modes using the following theorem:

Theorem 5-2-1 (LOC-77)

s_0 is a fixed mode of system (2-4-4), with simple modes, with respect to the control structure F if and only if s_0 is not a pole of any elementary circuit of graph D_F associated with the system.

This characterization is simple to use, and makes it easy to consider a general control structure. It may also be used to distinguish between two types of fixed modes: s_0 is a nonstructural fixed mode if s_0 is both a zero and a pole in the transmittance of one of the elementary circuits (pole-zero or zero-pole simplification), and it is structural if it does not belong to any typical transmittance polynomial of the elementary circuits of the graph. The drawback of this characterization is that it is valid only for systems with simple modes. It therefore must be expanded to cover any system.

Example 5-2-1

Let us reexamine example (3-4-1) whose modes were $\{a, 3, b, \frac{c \pm \sqrt{c^2+4}}{2}\}$;

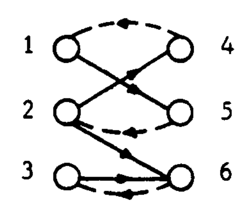
$F = \{(1,1),(2,2),(3,3)\}$

$V_{1F} = \{1,2,3\}$

$V_{2F} = \{4,5,6\}$

$E_{1F} = \{(1,5),(2,4),(2,6),(3,6)\}$

$E_{2F} = \{(4,1),(5,2),(6,3)\}$



The associated graph D_F is given by figure 5.1. It contains two elementary circuits which are:

(1) 1-5-2-4-1 with transmittance $t_1 = \frac{s-c}{(s^2-cs-1)(s-a)}$

(2) 3-6-3 with transmittance $t_2 = \frac{1}{s-3}$

The mode s=b is a structural fixed mode of the system because it does not belong to any typical polynomial of the elementary circuits of the graph. If a=c, the system will have a nonstructural fixed mode due to the simplification in the transmittance t_i.

V.3 CHARACTERIZATION USING GENERALIZED CACTUSES

A directed graph $D_S=(V,E)=(U \cup X \cup Y, E)$ is associated with the system (C,A,B) of (2-4-2) described by its state equation. The graph peaks V correspond respectively to inputs $U = \{u_1,...,u_m\}$, to states $X = \{x_1,...,x_n\}$ and to outputs $Y = \{y_1,..., y_p\}$. E is the arc set (V_j,V_i) directed from peak V_j to peak V_i, arcs (x_j,x_i) and $(x_j,y_i) \in E$ if and only if $a_{ij} \neq 0$, $b_{ij} \neq 0$ and $c_{ij} \neq 0$ respectively. In this section the following special subgraphs are defined: /82

Definition 5-3-1 (SIL-82b, PIC-83a)

1) INPUT PATH: this is either a path starting from an input peak (source) toward a state peak (well), or an input peak. It does not contain any output peak.

2) OUTPUT PATH: this is either a path starting from a state peak (source) toward an output peak (well), or toward an output peak (source). It does not contain any input peak.

3) INPUT-OUTPUT PATH: this is a path beginning from an input peak (source) toward an output peak (well).

4) STATE PATH: this is a path between two state peaks, or a single state peak. It does not contain any input or output peak.

5) CIRCUIT: already defined.

6) INPUT CACTUS: this is an input path with at least one state peak; the source and input cactus well are those of the input path. An input path connected to an elementary circuit is also an input cactus, except if it is connected to its well.

7) OUTPUT CACTUS: it's definition is similar to that of the input cactus.

8) CHAIN: this is a group of disjoined circuits connected together in a sequential manner, or a single circuit.

9) CONNECTION: this is an input path connected to the first circuit of a chain (but not to its well), an output path connected to the last circuit of a chain (but not to its source).

Figure 5.2 illustrates the different subgraphs given above:

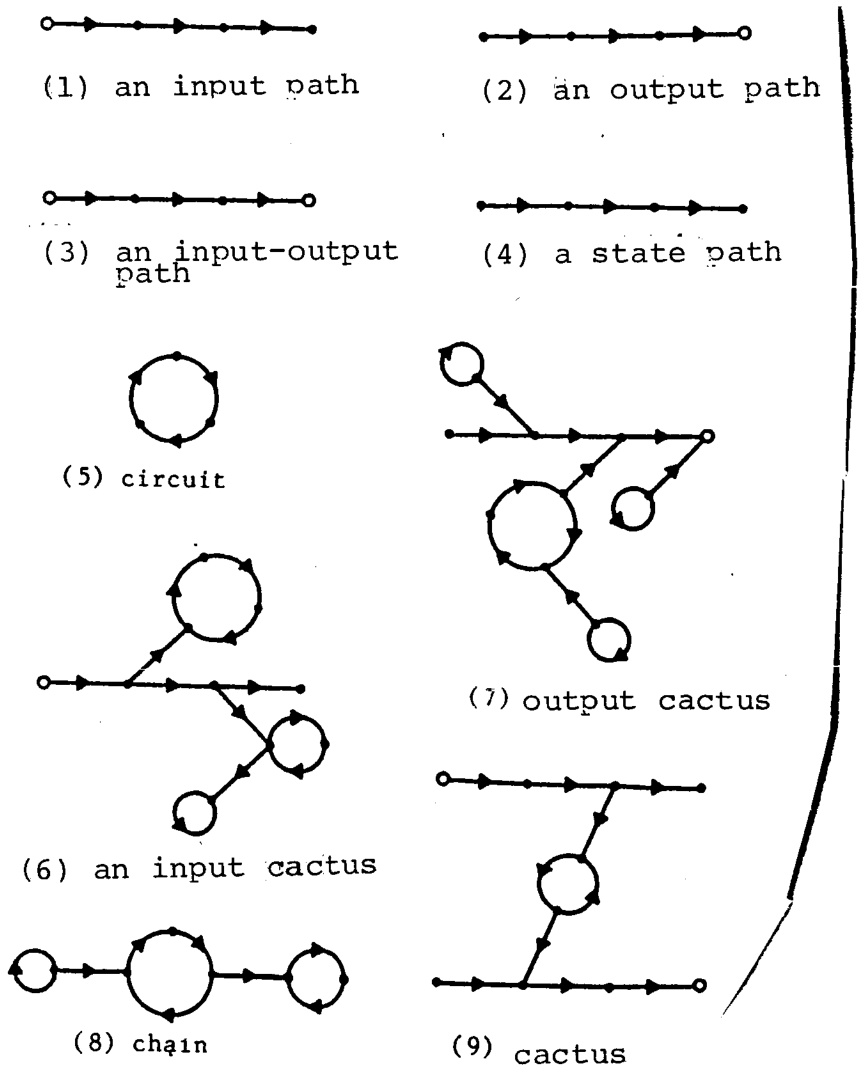


Fig. 5.2 - Special Subgraphs

Definition 5-3-2 (PIC-83a, SIL-82b) Generalized Cactus

A generalized cactus is defined as one of the following directed graphs:

i) The union of one (or more) input-output path(s) and an equal number of state paths, plus several possible circuits, input paths and output paths.

ii) The union of connections with (or without) several circuits, input paths and output paths.

iii) The union of input and output cactuses.

Picahi et al (PIC-83a) (SIL-82b) characterize structural fixed modes by the following theorem:

Theorem 5-3-1 (-83a)

The system (2-4-2) with N control stations has no structural fixed modes with respect to the F control structure if and only if the directed graph associated with each complementary subsystem (def. 2-4-7) is not a generalized cactus. /84

Pichai et al (PIC-83A) (SIL-82b) give an algorithm (for details on the algorithm, see (PIC-83a) or (SIL-82b) to test a graph made up of generalized cactuses. The algorithm must be used to test for the existence of structural fixed modes of a system with N control stations, for all complementary subsystems, i.e. 2^N-2 times. Consequently it leads to many binary calculations, in particular for a large system (large N) and this test is therefore impractical.

V.4 CHARACTERIZATION USING THE PROPERTY OF BEING HIGHLY CONNECTED

The system (2-4-2) and the set of control matrices K_F are taken into consideration. Given $D_S = (V,E)$ the directed graph associated with the system and defined in the previous paragraph. If control K_F is applied to the system, then the arcs corresponding to graph D_S are added. The graph associated with the closed loop system D_F becomes or E_F are the arcs corresponding to control K_F, i.e.:

$$E_F = \{ (y_j, u_i) : (k_f)ij \neq 0; \quad i=1,\ldots,m; \; j=1,\ldots,p\}$$

Linnemann (LIN-83) et PIchai et al (PIC-84) characterize the structural fixed modes with the following theorem:

Theorem 5-4-1 (LIN-83, FPIC-84)

The system (2-4-2) has no structural fixed modes with resepct to control structure K_F if and only if the following conditions are proved:

i) Each state peak, $x_i \in X$ $i=1,\ldots,n$, belong to a highly connected component of the graph associated with system D_F. The component should contain an arc $(y_j, u_i) \in E_F$.

ii) Disjoined circuits exist, $C_i = (V_i, E_i)$ $i=1,\ldots,q$ in D_F such that:

$$X \subset \bigcup_{i=1}^{q} V_i$$

This characterization uses the graph associated with the closed loop system, it is simple to use and makes it possible (see ch. VII) to determine an optimum control structure without fixed modes.

Example 5-4-1 /85

Given system (3-3-1), the graph associated with D_F is given by figure (5.3a).

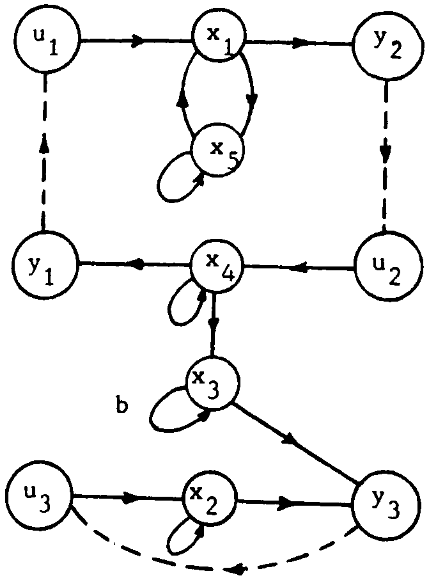


fig. 5.3a

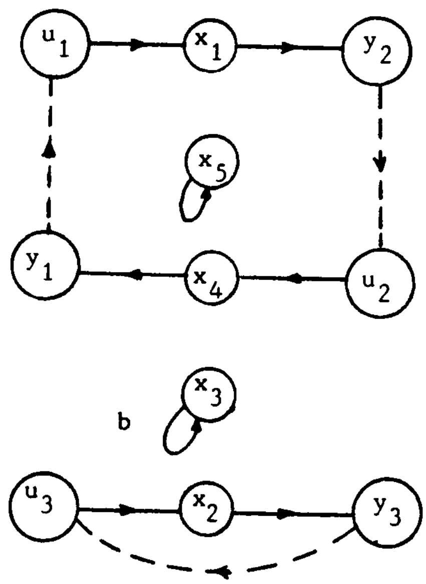


fig. 5.3b

Figure (5.3a) shows that x_3 is not contained in a highly connected circuit and the corresponding mode is therefore a type i structural fixed mode. Conversely, the system has no type ii structural fixed modes because disjoined circuits exist in the graph that contain all states (fig. 5.3b). If b = 0 we have a type ii structural fixed mode (fig. 5.3b) that proves condition i of theorem (5-4-1) (fig. 5.3a).

V.5 CHARACTERIZATION USING CIRCUIT FAMILIES

The square matrix M(s) with the following dimension (n+m+p) is associated with system (2-4-2):

$$M(s) = \begin{bmatrix} 0 & C & 0 \\ 0 & A-sI & B \\ K_F & 0 & 0 \end{bmatrix}$$

We have a directed graph D(M(s)) = (V,kE) with (n+m+p) peaks corresond to this matrix. Shown are output peaks notated $y_1, \ldots, y_p$, input peaks notated $u_1, \ldots, u_m$ and n state peaks notated $x_1, \ldots, x_n$. If element m_{ij} of M is not zero, arc $(j,i) \in E$ is directed from peak j to peak i and its weight is given by m_{ij}.

Definition 5-5-1 (REI-83A) (REI-84a) Circuit Family

The circuit set with disjoined peaks is called a circuit family. It is characterized by its dimension. The family weight is the weight of all arcs in the family, and the dimension n_X of a family is given by the number of state peaks belonging to the family.

With these definitions Reinschke (REI-83) (REI-84a) (REI-84Ab) proposes a fixed mode characterization, based on a graphic interpretation of the factors of the typical closed loop polynomial given by the following theorem:

Theorem 5-5-1 (REI-84a)

The coefficients a_i $1 \leq i \leq n$ of the typical closed loop polynomial:

$$\det(sI-A-BK_FC) = s^n + a_1 s^{n-1} + \dots + a_i s^{n-i} + \dots + a_{n-1} s + a_n \tag{5-5-1}$$

are determined by circuit families of dimension i of graph D(M(s=0)), each circuit family of dimension i corresponds to a term a_i^j of the sum $a_i = \sum_j a_i^j$, the numerical value of this term is given by the weight of the corresponding circuit family, and its sign is given by a sign factor equal to $(-1)^d$ where d is the number of disjoined circuits of the family under study.

In particular:

a_1: is obtained based on all circuits of dimension 1, and the sign factor is equal to (-1).

a_2: is obtained from all circuits of dimension 2, each with a factor whose sign is equal to (-1), and all pairs of disjoined circuits of dimension 1, each with a sign factor equal to (+1).

a_3: is obtained based on all circuits of dimension 3 (each with a sign factor equal to (-1), all pairs of a circuit of dimension 2 and a circuit of dimension 1 (each pair has a sign factor equal to +1), and all triplets of disjoined circuits of dimension 1 (each triplet has a sign factor equal to (-1).

Using theorem (5-5-1) Reinschke (REI-83) (REI-84a) characterizes the fixed modes with the following theorem:

Theorem 5-5a-2 (REI-84)

s_0 is a multiple fixed mode of order h of the system with respect to control K_F if and only if for $j = n, n-1, \ldots, n-h+1$ ($j \neq n-h$) one of the two following conditions is verified:

i) There is no circuit family of dimension j in the graph $D(M(s_0))$. /87

ii) There are two (or more) circuit families of dimension j in $D((M(s_0))$ which numerically simplify each other for all permissible values of the control matrix.

Reinskhe (REI-84b) showed that for type ii structural fixed modes (at the origin), it is condition i that is verified. In general, it is possible to determine whether a structural fixed mode or nonstructural fixed mode is involved, and if this is the case to determine the conditions for which the system has nonstructural fixed modes by calculating the coefficients a_n of the typical polynomial.

Example 5-5-1

Given system (3-3-1), with a = 2, b = 4 and c = 1. graph $G(M(s))$ corresponding to matrix $M(s)$ (for an unspecified s and a decentralized control $K = \text{diag}(k_{11}, k_{22}, k_{33})$) associated with the system is given by figure (5.4).

In graph G(M(s)) there are six circuit families of dimension 5 that are given by table (5.1).

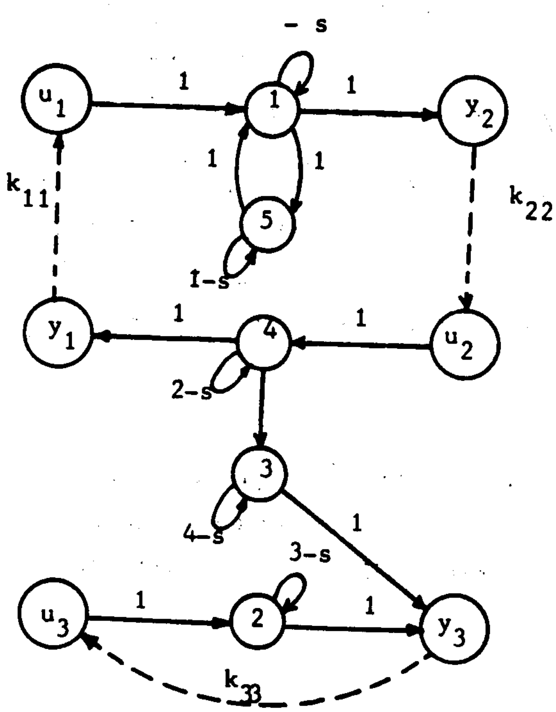
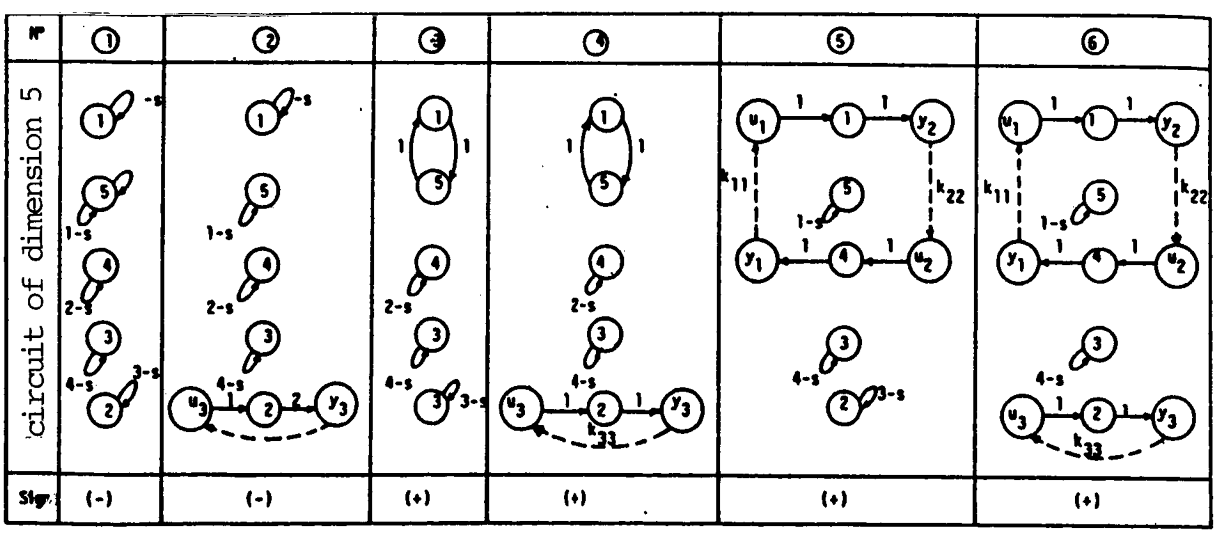


Fig. 5.4 (Graph G(M(s)))

The typical polynomial of the system is $\Phi(s) = (s-2)\ (s-4)\ (s-3)\ (s^2-s-1)$. If we consider $s = 4$, all families of Tab. 5.1 are reduced to families of dimension 4, then $s = 4$ is a fixed mode. For $s = 2$, families 1, 2, 3 & 4 disappear and we have two families of dimension 5 that give coefficient a_5 of the typical polynomial of a closed loop system (5-5-1):

$$a_5 = k_{11}\ k_{22}\ (1-2)\ (4-2)\ (3-2+k_{22}) =$$

$$-\ 2\ k_{11}\ k_{22}\ (1+k_{33}) \neq 0$$



Tab. 5.1 - Circuit Families of Dimension 5
of Graph in Fig. 5.

therefore s = 2 is not a decentralized fixed mode of the system. If the numerical values of A, B and C are replaced by arbitrary elements, we have:

$$a_5 = (a_{33}-s)\,\{a_{15}\,a_{51}\,(a_{44}-s) + b_{11}\,c_{21}\,k_{22}\,b_{42}\,c_{14}\,k_{11}\,(a_{55}-s)\}\,b_{25}\,c_{33}\,k_{33}$$

this factor is zero if $s = a_{33}$ ($\forall\, a_{ij}$, b_{ij} $c_{ij} \longrightarrow s$ is therefore a structural fixed mode) or if $a_{44} = a_{55}$ (the equality between elements a_{44} and a_{55} of A reveals a nonstructural fixed mode in $s = a_{44} = a_{55}$).

V.6 COMMENTS

The graphic characterizations shown in this chapter use a graph associated either with a closed loop system (PIC-83a), or a closed loop system (LOC-77) (LIN-83) (PIC-84) (REI-83).

Pichai et al's approach (PIC-83a) using a graph associated /89
with an open loop system, consists of testing the properties of 2_N-2 graphs associated with complementary subsystems of the system. Consequently, there are many binary calculations and the computer time is long. This test is impractical and is only valid for structural fixed modes.

The approach using a graph associated with a closed loop system is more interesting that the preceding one, because it shows an overview of the system. The available methods are broken down into two groups depending on whether the graph associated with the system either:

-By its transfer matrix (a peak represents an input or an output). The only available melthod in this group is that of Locatelli et al (LOC-77); it is very interesting and simple and may be used to specify the origin of the fixed modes, but it is only valid for systems with simple modes. This method deserves to be expanded to cover all cases.

-By its state equation (a peak represents an input, a state or an output) (LIN-83) (PIC-84) (REI-83). Linnemann's (LIN-83) and Pichai et al's (PIC-84) method is simple, purely graphical, but it is only valid for structural fixed modes. Reinschke's method (REI-83) is general, valid for mode types, but requires considerable calculations.

Note that the closed loop approach makes it easier to include any control structure.

CHAPTER VI - STABILIZATION OF DECENTRALIZED SYSTEMS IN THE PRESENCE OF FIXED MODES

VI.1 INTRODUCTION

We have shown in previous chapters that fixed modes may be structurally controllable or structurally uncontrollable under structural stresses. The appearance fixed modes that are controllable in a decentralized manner is either due to the perfect inequality of certain system parameters or to the multiplicity of the modes (nonstructural fixed modes), or because the position of the mode at the origin makes it independent of the system parameters (type ii_2 structural fixed modes). However the existence of structurally fixed modes that are uncontrollable under decentralized conditions is due to the lack of information transfer between the subystems (type i and ii_1 structural fixed modes).

In this chapter we shall focus on the stabilization of systems /92
with fixed modes that are structurally controllable under structural stresses, and show that it is possible to stabilize the system while maintaining the stress over the structure by changing the control structure. This is an essential difference between centralized and decentralized fixed modes. In effect, when the system has a centralized fixed mode (uncontrollable and or unobservable mode) in s_0, then for any control law used (linear or not, dynamic or not, distributed or not, of finite dimension or

not) the fixed mode will remain in that the feedback of the closed loop system will always have a term proportional to exp ($s_0(t)$. However Wang and Davison (WAN73-b) showed that a decentralized fixed mode remains so, even when using a dynamic control. Furthermore, Kobayashi (KOB-78) and Anderson and Moore (AND-81b) showed that a general control law in the form:

$$u_i(t) = K_i\ (I_i(t)) \qquad i=1,\dots,N \tag{6-1-1}$$

where $I_i(t) = \{y_i(q),\ u_i(r);\ q \in [o,t] \text{ and } r \in [o,t]\}$ is the information available at station i, may stabilize systems structurally fixed modes that are controllable under decentralized conditions in the sense that this law (6-1-1) may transfer the system from an initial state to the origin in a finite period of time, because the highly connected property of the system transfer matrix allows a control station to transfer information through the state space to another station by using signalling strategies (KOB-82).

Consequently, system stabilization in the presence of structurally controllable fixed modes is feasible if one of the properties of linearity or invariance with time is sacrificed. This result is easy to interpret using Anderson's fixed mode characterization (AND-82) (see III.4.6). For simplification, let us consider a system with two single input/single output subsystems:

$$G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \tag{6-1-2}$$

By applying a control $u_2 = k_2\ y_2$ to the 2nd subsystem, the transfer function seen from the first station is:

$$\frac{y_1(s)}{u_1(s)} = g_{11}(s) + g_{12}(s) \cdot \frac{k_2}{1-g_{22}(s)k_2} \cdot g_{21}(s) \tag{6-1-3}$$

If s_0 is a simple mode of the system, theorem (3-4-11) shows that a fixed mode decentralized in s_0 appears if and only if g_{12} has a pole in s_0 and g_{21} has a zero in s_0. In other words, the appearance of a simple fixed mode is due to the zero-pole (or pole-zero) simplification in product $g_{12} \frac{k_2}{1-g_{22}\, k_2} g_{21}$. However if k_2 is not linear or is time-variant, /93 then this simplification is no longer feasible, since a time-variant block and an adjacent invariant block cannot be switched. Thus there is no longer any juxtaposition of a pole-zero pair that are simplified together. In the case of a multiple mode the simplification is done between a g_{12} (or g_{21}) pole and a g_{22} pole $\left(\frac{k_2}{1-g_{22}k_2}\right)$, zero), using a time-variant control also prevents this simplification and makes stabilization feasible. Note that if one of the transfer functions between stations is zero (block-triangular transfer matrix), then the system is structurally controllable under the structural stress, and the only way to control it is to relieve the stresses on the control structure (see ch. VII).

The systems considered here have N stations and are described by (2-4-1) or globally by (2-4-2).

VI.2 USING A SAMPLING/BLOCKING UNIT

Let us consider system (C,A,BO of (2-4-1) and let us put a sampling-blocking unit, of order $a \neq 0$ and with a sampling cycle T, in a row at the input of each system station. The resulting discrete system is:

$$X((k+1)T) = Q\, X(kT) + \sum_{i=1}^{N} R_i\, u_i(kT)$$
$$y_i(kT) = C_i\, X(kT) \qquad (6\text{-}2\text{-}1)$$

with $Q = \exp\,(AT)$

$$R_i = \exp\,(-\,aT) \int_0^T \exp\,(A+aI)dt\; B_i$$

The decentralized fixed modes of the discrete system (6-2-1) are given by (WAN-82):

$$\bigcap_{K_i \in R^{m_i x p_i}} \sigma(Q + \sum_{i=1}^{N} C_i K_i R_i) \qquad (6\text{-}2\text{-}2)$$

Theorem 6-2-1 (WAN-82)

System (2-4-1) is stabilizable by local discrete controls if a sampling period T and a real a such that the fixed modes of the discrete system (6-2-1) are in the unity circle.

Wang (WAN-82) thus proposes a sufficient condition for the /94
existence of a decentralized discrete control, but he does not provide any constructive procedure for characterizing the sampler (T and a). All that we know is that the sampler should be of a general type ($a \neq 0$), i.e.:

$$u(t) = \exp\{-a(t-kT)\}\, u(kT) \quad kT \leq t < (k+1)T$$

as a zero order blocking unit may be insufficient, particularly if the fixed modes are at the origin (see remark 1 in WAN-82).

Note that the use of a discrete decentralized control for stabilizing the system is valid only if the fixed modes are simple: let us assume that s_0 is a fixed mode of the system, if s_0 is simple, the transfer functions of the local subsystems do not have a pole in s_0 and therefore by quantifying the inputs of the complementary subsystems (g_{12} and g_{12} see 6-1-3 and fig. 3.2b). If s_0 is multiple then one of the transfer functions of the local subsystems (g_{11} or g_{22} see 6-1-3) has a pole in s_0 (AND-82), and therefore the simplification is done between a pole of one of the complementary subsystems (g_{12} or g_{21}) and a pole of one of the local subsystems (g_{11} or g_{22}). The quantification no longer prevents simplification and the discrete system will always have a fixed mode in s_0, see fig. 6.1.

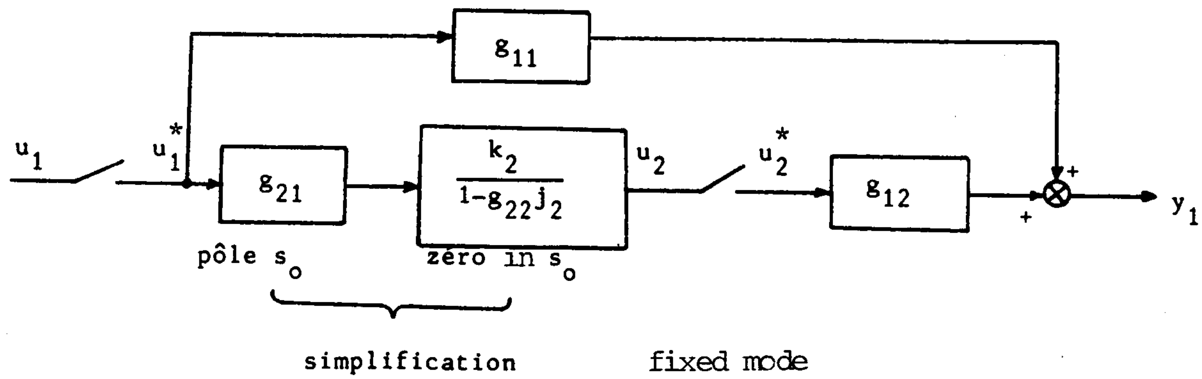


Fig. 6.1 - Quantification of System Inputs
Multiple Fixed Mode

VI.3 USE OF A TIME-VARIANT CONTROL

This section shows Anderson's and Moore's results (ABND-81b) and Purviance's and Tylee's results (PUR-82) which use a decentralized control, variant with time, for stabilizing the linear time-invariant systems, in the presence of fixed modes. First, let us show a special case of systems with two stations, then let us briefly discuss how we can apply Anderson's and Moore's results to the general case of N subsystems.

The system is considered to be controllable and observable /95
with two control and observation stations:

$$\begin{aligned} \overset{\circ}{X}(t) &= A\,X(t) + B_1\,u_1(t) + B_2\,u_2(t) \\ y_i(t) &= C_i\,X(t) \qquad i=1,2 \end{aligned} \tag{6-3-1}$$

where $x \in R^n$, $u_i \in R^{m_i}$ & $y_i \in R^{p_i}$. Let us apply a time-variant control, (where time is periodic with period T) to the 2nd station:

$$u_2(t) = K_2(t)\;y_2(t) \tag{6-3-2}$$

we obtain:

$$\begin{aligned} \overset{\circ}{X}(t) &= A + B_2\,K_2(t)\,C_2 + B_1\,u_1(t) \\ y_1 &= C_1\,X(t) \end{aligned} \tag{6-3-3}$$

Let $\Phi_{K_2}(t,s)$ be the transition matrix of the looped system (6-3-3), then the observabililty and controllability systems are:

$$GO(s,s+T) = \int_{s}^{s+T} \Phi_{K_2}^{T}(t,s)\ C_1^T\ C_1\ \Phi_{K_2}(t,s)\ dt \qquad (6\text{-}3\text{-}4)$$

$$GC(s,s+T) = \int_{s}^{s+T} \Phi_{K_2}(s,t)\ B_1\ B_1^T\ \Phi_{K_2}^{T}(s,t)\ dt \qquad (6\text{-}3\text{-}5)$$

The system is said to be uniformly controllable (observable) if the controllability system GC(s,s+T) (observability GO(s,s+T) is strictly defined to be positive. Satisfaction of the uniform controllability and observability properties implies that stabilization of a closed loop system (6-3-3) is possible by a linear state feedback.

VI.3.1. Piece-Wise Constant Control

Anderson and Moore (AND-81b) propose the use of a time-variant control that is periodic, piece-wise constant for stabilizing decentralized systems in the presence of fixed modes. With the assumption of overall controllability and observability and the assumption of the highly connected property of complementary subsystems, i.e.:

$$G_{12}(s) = C_1(sI-A)^{-1}\ B_2 \neq 0 \qquad (6\text{-}3\text{-}6)$$

$$G_{21}(s) = C_2(sI-A)^{-1}\ B_1 \neq 0 \qquad (6\text{-}3\text{-}7)$$

they propose the following theorem in the form of two lemmas:

Theorem 6-3-1 (AND-81b)

Given system (6-3-1) (controllable and observable), then it may be made uniformly controllable and uniformly observable for station 1 by applying a time-variant control to station 2: $u_2(t) = K_2(t)\ y_2(t)$, or $K_2(t)$ is periodic, with an arbitrary period, and piece-wise constant taking at least 1 + max (m_2, p_2) distinct values in one period. /96

Note: the assumptions required by the theorem are equivalent to the assumption of structural controllability and observability under decentralization stresses.

This result is analyzed as follows: when the fixed mode is due to a lack of observability from station 1, station 2 observes the fixed mode (overall observability assumption) and the return to station 1 through transmission channel $G_{12} \neq 0$. When the fixed mode is due to a lack of controllability for station 1, then G_{21} transfers the controls from station 1 to the input of station 2. Thus these signals assign modes that are not directly accessibly by station 2 (see fig. 3.2).

Anderson and Moore apply their study to the case of system with N stations. They show that with the assumptions of highly connected complementary subystems $G_{ij} \neq 0$ (structurally controllable observable system under structural stress), the system can be made controllable and observable by a station by successively applying time-variant controls $u_i(t) = K_i(t)\ y_i(t)$ $i=2,\ldots,N$. Each control $K_i(t)$ is periodic, with period T, and piece-wise constant. For each station i, there are as many invariant systems to consider as there are distinct $K_{i-1}(t)$ values, which gives a gain $K_i(t)$ taking at least $\prod_{j=2}^{i} \max(m_j, p_j) + 1$ distinct values in a period. This rapidly increases if systems with a large number of stations are considered, and if it is taken into account how difficult it is to implement such a control.

Example 6-3-1 (AND-81b)

Let us consider the following controllable and observable system (with two stations):

$$\overset{\circ}{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} X + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

$$y_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} X$$

$$y_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} X$$

The system has a nonstructural decentralized fixed mode in s = 1. let us apply a control, variant with time, to station 2: $u_2(t) = K_2(t)\ y_2(t)$ with:

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$$K_2(t) = \begin{cases} [0 \quad 1] & \text{for } t \in [2k,\ 2k+1] \\ [1 \quad 0] & \text{for } t \in [2k+1,\ 2k+2] \end{cases} \qquad k=0,1,2,...$$

We obtain the following in a closed loop:

$$\mathring{X}(t) = A + B_2\ K_2(t)\ C_2\ \ X(t) + B_1\ u_1(t)$$
$$y_1(t) = C_1\ \ X(t)$$

The controllability and observability properties of a closed loop system, calculated analytically on interval [2k, 2k+2] for k=0,1,2,... are defined to be positive with a conditioning number * of 6 and 100 respectively. We therefore obtain a reasonable controllability. However, the observability may be improved by selecting another type of time-variant control, as we shall see in the next section.

VI.3.2 Sinusoidal Control

Purviance and Tylee (PUR-82) considered the special case of systems with two single input/single output subsystems, having a decentralized fixed mode. They offset the lack of observability by station 1 (cause for the existence of a fixed mode) by communicating to this station the fixed mode value (which is observable from station 2) through the transmission channel G_{12}, via a time-variant feedback $K_2(t)$. This is a well known theory of communication problem, and a good solution is to use sinusoidal modulations, with a frequency comparable to the response frequency of the communication channel G_{12} (VAN-68).

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*The conditioning number (CN) of a rectangular matrix Q of full rank is given by the ratio between the highest and lowest singular value of the matrix (MOO-81). CN is a good measurement of the robustness of rank Q. If CN = 1, the effective rank of Q is equal to the nominal rank ...[page cut off].

Example 6-3-2 (PUR-82)

Let us consider the following system with two inputs and two outputs:

$$G(s) = \left[\begin{array}{c|c} 0 & \frac{s}{(s+1)(s+2)} \\ \hline \frac{1}{s} & 0 \end{array}\right]$$

It is clear that the system has a type ii structural decentralized fixed mode. let us apply a sinusoidal control u_2 = k sin
The conditioning number of the matrix of the observability property of the looped system is given by figures (6.1a) and (6.1b) for two k amplitudes and different values of frequency

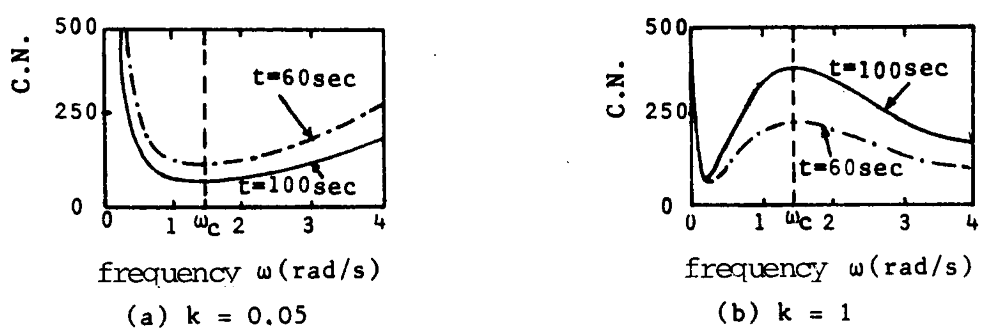

Fig. 6.1 - Conditioning Number
of Observability Matrix of Example 6-3-2

Note that for $\omega = \omega_c = \sqrt{2}$ (ω_c is the center frequency of the passband filter G12 through which the transmission is done). An optimum communication can be expected between the two stations. The conditioning number of the observability property is minimum for k = 0.05 (for t = 100, 6.1b gives CN $\simeq$ 400), which shows the significance of the energy selected of the signal to be transferred through the feedback loop. Actually, an increase in this energy destroys the balance between the modes of the system and this leads to a high conditioning number and therefore to poor observability.

To compare with Anderson's and Moore's result (AND-81b), let us apply a periodic control, of period 2, piece-wise control to the 2nd station with:

$$K_2(t) = \begin{cases} 0 & 0 \leqslant t < 1 \\ 1 & 1 \leqslant t < 2 \end{cases}$$

The conditioning number of the observability property GO(100.0) is equal to 132.9. /99

In short, use of a sinusoidal control (low-energy) for this example decreases the conditioning number of the observability property by 54% relative to the binary control proposed by Anderson and Moore. This therefore provides better observability. Similar results are obtained for controllability.

VI.3.3 Comments

Using a time-variant decentralized control, we showed that fixed modes, structurally controllable under decentralization stresses, can be stabilized. The advantage of this approach is to keep a decentralized structure of the control. The difficulty in making piece-wise constant controls makes them impractical in practice, which is not the case of the sinusoidal control, but the results available for this approach cover only a simple two system single input/single output case. These results must therefore be applied to the case of N control stations, a case which is of great importance for practical applications. Such a study will be presented in the next section using the "vibrational control".

VI.4 USING A VIBRATIONAL CONTROL

Meerkov (MEE-73) introduced the principle of vibrational control in 1973, and showed that by introducing vibrations in the parameters of a system, it is possible to stabilize the unstable modes of the system for which traditional control methods (output or state feedback, anticipation) are not applicable because measurements of the system are lacking (unmeasurable variable or variable difficult to measure).

In effect, the advantage of a vibrational control over a traditional control is that it does not require measurements on the system, because it simply consists of introducing high frequency vibrations into the dynamic matrix of the system. These vibrations do not depend on the state (no feedback) any more than on additive control signals (no anticipation), but depend only on time.

If we consider a structural stress on a control as a lack of measurement at the local level, then it seems conceivable to use a vibrational control to stabilize the unstable fixed modes of the system when a traditional control fails.

VI.4.1. Principle of a Vibrational Control /100

A vibrational control consists of introducing vibrations (oscillations), with a zero mean value, on the system parameters, to modify its dynamic behavior in the desired direction, particularly to stabilize it. These vibrations are directly introduced into the dynamic matrix of the system by oscillating the technological parameters of the system (for example by varying the amplification factors). This control does not require any measurements of the system (no need for control or observation matrices). Accordingly the synthesis of the control appears in the same way as for systems whether or not they are self-contained.

Let us assume that we have a linear, self-contained system that is time-invariant:

$$\overset{\circ}{X}(t) = A\,X(t) \qquad A = \| a_{ij} \|^{n}_{i,j=1} \text{ and } \quad X \in R^n \qquad (6\text{-}4\text{-}1)$$

Let us introduce periodic vibrations into the system that have a zero mean value and expressed as follows:

$$V(t) = \|v_{ij}(t)\|^{n}_{i,j=1} \tag{6-4-2}$$

with $\quad v_{ij}(t) = q_{ij} \sin r_{ij}t \qquad r_{ij} \neq r_{s\ell}$ for $ij \neq s\ell$

We obtain: :

$$\mathring{X}(t) = \left[A + V(t)\right] X(t) \tag{6-4-3}$$

Before analyzing the time-variant system (6-4-3), a question is posed: When can a vibrational control be applied?

Definition 6-4-1 (MEE-80)

The system (6-4-1) is said to be stabilizable vibrational if a periodic matrix V(t) with a zero mean value exists such that system (6-4-3) is asymptotically stable.

Theorem 6-4-1 (MEE-80)

It is assumed that a line vector c exists so that the pair (c,A) is observable. Therefore a necessary and sufficient condition for system (6-4-1) to be vibrationally stabilizable is that the outline of its dynamic matrix is negative (Tr A < 0).

An analysis of the time-variant system (6-4-3) is based on Volosov's (MEE-73) (MEE-80) procedure for finding the mean value which imposes certain conditions on the vibrations matrix V(t) (quasi-triangular structure)*, sufficiently large amplitudes and frequencies** and provides the invariant system

$$\mathring{Z}(t) = \left[A + \overline{V}\right] Z(t) \tag{6-4-4}$$ /101

where $\overline{V}$ is a constant matrix dependent on the vibrational amplitudes and frequencies). This system (6-4-4) describes the variant system (6-4-3) "on the average", if the system (6-4-4) is stable, system (6-4-3) will also be stable. The parameters of the invariant system (6-4-4) depend on the q_{ij}, r_{ij} and a_{ij};

---

* because we do not know how to analyze the system if V(t) has another structure (MEE-80).
**See theorem (6-4-2).

using the stability criteria of invariant systems (Routh's criterion for example) the stability conditions of this system can be determined and therefore the amplitudes q_{ij} and frequencies r_{ij} of the vibrations $V(t)$ that stabilize the variant system (6-4-3).

Given the lower (or upper since we can go from one to the other) quasi-triangular vibrations matrix:

$$V(t) = \begin{bmatrix} 0 & & & & & \\ q_{21} \sin r_{21} t & 0 & & & \mathbb{O} & \\ q_{31} \sin r_{31} t & q_{32} \sin r_{32} t & 0 & \cdots & & \\ \vdots & & & \ddots & 0 & \\ q_{n1} \sin r_{n1} t & q_{n2} \sin r_{n2} t) & \cdots & & q_{n,n-1} \sin r_{n,n-1} t & 0 \end{bmatrix} \qquad (6\text{-}4\text{-}5)$$

Matrix $\bar{V}$ (and therefore the invariant system (6-4-4)) is determined as follows:

Let us consider the system of differential equations:

$$\overset{o}{X}(t) = V(t)\, X(t) \qquad X \in R^n \qquad (6\text{-}4\text{-}6)$$

where $V(t)$ is defined in (6-4-5). Let us assume that the initial conditions of (6-4-6) are given by: $x_i(t_o) = x_i^o \quad i=1,\ldots,n.$ Therefore the solutions of the first two equations of (6-4-6) are:

$$x_1(t) = x_1^o$$

$$x_2(t) = x_2^o + \left[F_{21}(t) - F_{21}(t_o)\right] x_1^o$$

where F_{21} (t) is a periodic function of the zero average time. Now let us determine the solution of the 3rd equation of (6-4-6) by letting $F_{21}(t_0) = 0$. A similar procedure is applied successively to solve the remaining equations of (6-4-4). If we find the following while solving equation i:

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$$x_k = x_k^o + \sum_{j=1}^{k-1} \left[F_{kj}(t) - F_{kj}(t_o)\right] x_j^o \qquad k=1,\ldots,i-1$$

then the solution for equation i of (6-4-6) is determined by substituting

$$x_k = x_k^o + \sum_{j=1}^{k-1} F_{kj}(t)\, x_j^o \qquad k=1,\ldots,i-1$$

We therefore obtain the matrix $F(t) = \| F_{ij}(t) \|_{i,j=1}^{n}$ where $F_{ij}(t)$ are periodic zero average functions. Let us define matrix E so that:

$$E = \| e_{ij} \|_{i,j=1}^{n} \text{ and } e_{ij} = \lim_{T \to \infty} \frac{1}{T} \int_o^T F_{ij}^2(t)\, dt \qquad (6\text{-}4\text{-}7)$$

Let us note that matrix E has the same structure as as V(t). Finally, matrix V of equation (6-4-4) is given by:

$$\overline{V} = -(A^T \odot E) \qquad (6\text{-}4\text{-}8)$$

where $\odot$ represents the element by element multiplication of matrices A^T and E, i.e.:

$$\overline{v}_{ij} = -a_{ij} \cdot e_{ij}$$

Theorem 6-4-2 (MEE-80)

Constants Q_0 and R_0 exist and they are large enough so that if $q_{ij} \gg Q_0$ and $r_{ij} \gg R_0$ a for each i,j, the X(t) solutions of (6-4-3 and 6-4-5) and Z(t) of (6-4-4 and 6-4-8) (defined with identical initial conditions X(o) = Z(o)) are related by the expressions:

$$\overline{X}(t) = [I + F(t)]\, Z(t) \qquad t \in [0, \infty[$$
$$\| X(t) - \overline{X}(t) \| < \frac{1}{Q_o} \qquad (6\text{-}4\text{-}9)$$

where I is the unity matrix and F(t) the matrix calculated by the preceding procedure. If system $\dot{Z}(t) = [(A+\overline{V})] Z(t)$ is asymptotically stable, relation (6-4-9) is verified for $t \in [0,\infty[$; otherwise it is verified only for $t \in [0, Q_0[$.

The theorem shows that if the vibrational amplitudes and frequencies are high enough, then system (6-4-4 and 6-4-8) describe on the average the trajectories of the time-variant system (6-4-3 and 6-4-5) and therefore all properties of the system (6-4-4 and 6-4-8) are on the average true for the time-variant system (6-4-3 and 6-4-5). In particular if (6-4-4 and 6-4-8) is asymptotically stable then (6-4-3 and 6-4-5) is also stable.

According to (6-4-8), it is evident that the vibrations only have an effect on the system dynamics if $\overline{V} \neq 0$. Otherwise, the dynamics of the variant system (6-4-3) is the same as the dynamics /103 of the invariant system (6-4-4). Thus the vibrationally controllable elements a_{ij} of A are those for which $i > j$ and $a_{ij} \neq 0$ (for the selection of the lower triangular form).

Since the introduction of vibration deds makes the technological parameters of the system oscillate, the introduction of vibrations on zero elements of matrix A is not technologically feasible. Consequently, vibrationally controllable elements a_{ij} are nonzero elements $a_{ij} \neq$ of A for which $i > j$ and $a_{ij} \neq 0$.

Remarks:

1 - To establish an analogy between a vibrational control and a traditional control, note that the introduction of vibrations V(t) on system elements has a similar effect to the output feedback control characterized by $\overline{V}$ = BKC, or to a time-variant output feedback control and characterized by V(t) = B K(t) C.

2 - In some cases (see MEE-73), we can relief the stresses on

the vibrational amplitudes, i.e. not account for a "high enough q_{ij} condition" and be content with small amplitudes.

3 - If the system is not self-contained, the stabilization method using a vibrational control does not differ from that of the self-contained systems described above. Given the following system: $\overset{\circ}{X} = A X + B U$ $X \in R^n$, $U \in R^m$. It is assumed that the system controls are slow functions of time in that $|\frac{du_i}{dt}|$ is limited. In this case, the following supplementary condition must be added to the vibrations frequency:

$$r_{ij} \gg \max_{\ell} (\sup \frac{du_\ell}{dt}) \quad i,j=1,\ldots,n \quad \& \quad \ell=1,\ldots,m.$$

4 - A calculation of the $\bar{V}$ elements involves the integration of trigonometric terms that are difficult to develop. In the special case where we have a single element vibrationally controllable per line in matrix A, then the calculation becomes relatively simple. In this case we have:

$$v_{ij}(t) = q_{ij} \sin r_{ij} t \qquad F_{ij}(t) = - \frac{q_{ij}}{r_{ij}} \cos r_{ij} t$$

which gives:

$$e_{ij} = \frac{q_{ij}^2}{2 r_{ij}^2}$$

and:

$$\bar{v}_{ij} = - a_{ij} \cdot e_{ij} = - a_{ij} \frac{q_{ij}^2}{2 r_{ij}^2} \qquad (6\text{-}4\text{-}10)$$

VI.4.2. Stabilization Using a Vibrational Control /104

In this section we will consider the system class which has unstable fixed modes with respect to a given structural stress and whose dynamic matrix verifies the conditions of theorem (6-4-1). Vibrations are directly applied to the self-contained system shown below:

Example 6-4-1

Let us consider the following system with two subsystems:

$$\overset{\circ}{X} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} X + \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_2$$

$$y_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} X \qquad (6\text{-}4\text{-}11)$$

$$y_2 = \begin{bmatrix} 3 & -1 & 0 \end{bmatrix} X$$

It is easy to check whether the system has an unstable decentralized fixed mode in s=1. Given a self-contained system associated with system (6-4-11):

$$\overset{\circ}{X} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} X$$

This system is vibrationally stabilizable, since for c = (1 0 0) the pair (c,A) is observable, and Tr = - 2 < 0.

The vibrationally controllable elements are a_{21} and a_{32} if a quasi-triangular lower vibrations matrix is selected, or a_{12} and a_{23} if a higher quasi-triangular structure is selected.

Given in the first case:

$$V(t) = \begin{bmatrix} 0 & 0 & 0 \\ q_{21} \sin r_{21} t & 0 & 0 \\ 0 & q_{32} \sin r_{32} t & 0 \end{bmatrix}$$

To apply Volosov's procedure for establishing an average, let us consider a self-contained system associated with vibrations:

$$\overset{\circ}{X} = V(t)\, X \qquad \text{with} \qquad X(o) = X^{o}$$

the solution of this system is given by:

1st equation: $\quad x_1(t) = x_1^{o}$

2nd equation: $\quad \overset{\circ}{x}_2(t) = q_{21} \sin r_{21} t\, x_1^{o}$

$$x_2(t) = x_2^{o} + \left[F_{21}(t) - F_{21}(t_o)\right] x_1^{o}$$

with $\quad F_{21}(t) = -\dfrac{q_{21}}{r_{21}} \text{cost}\, r_{21} t$

3rd equation: $\overset{o}{x}_3(t) = q_{32} \sin r_{32} t \; x_2(t)$

Let us replace $x_2(t)$ by letting $F_{21}(t_o) = 0$, after integration we obtain:

$$x_3(t) = x_3^o + \left[F_{32}(t) - F_{32}(t_o)\right] x_2^o + \left[F_{31}(t) - F_{31}(t_o)\right] x_1^o$$

with $F_{32}(t) = -\dfrac{q_{32}}{r_{32}} \cos r_{32} t$

Let us determine matrix E (see 6-4-7):

$$e_{21} = \lim_{T \to \infty} \frac{1}{T} \int_o^T F_{21}(t)\, dt = \lim_{T \to \infty} \frac{q_{21}^2}{2r_{21}^2} \left\{ \frac{\sin 2 r_{21} T}{2 r_{21} T} + 1 \right\} = \frac{q_{21}^2}{2 r_{21}^2}$$

We obtain:

$$e_{32} = \frac{q_{32}^2}{2 r_{32}^2}$$

According to (6-4-8) we have

$$\bar{v}_{21} = - a_{12} \cdot e_{21} = - \frac{q_{21}^2}{2 r_{21}^2} = - e_{21} < 0 \qquad (6\text{-}4\text{-}12)$$

$$\bar{v}_{32} = - a_{23} \cdot e_{32} = - \frac{q_{32}^2}{2 r_{32}^2} = - e_{32} < 0$$

this gives us the time-invariant system, which on the average describes the variant system

$\overset{o}{X} = \left[A + V(t)\right] X$:

$$\overset{o}{Z} = (A+\bar{V})\, Z = \begin{bmatrix} -2 & 1 & 0 \\ 1+\bar{v}_{21} & 1 & 1 \\ 1 & -1+\bar{v}_{32} & -1 \end{bmatrix}$$

Let us set the stability conditions of this system using Routh's criterion:

$$\det(sI-A-V) = s^3 + 2 s^2 - (1 + \bar{v}_{21} + \bar{v}_{32}) s - 2 - \bar{v}_{21} - 2 \bar{v}_{32}$$
$$= a_3 s^3 + a_2 s^2 + a_1 s + a_o$$

The stability conditions are given by:

$$a_i > 0 \qquad i = 0,1,2,3$$

$$a_1 a_2 - a_o a_3 > 0$$

We therefore obtain: $\bar{v}_{21} < -2(1+\bar{v}_{32})$

$\bar{v}_{21} < 0$

Let us add conditions (6-4-12), we obtain: /106

$$\bar{v}_{21} < 0$$
$$\bar{v}_{32} < 0$$
$$\bar{v}_{21} < -2(1+\bar{v}_{32})$$

The range D of acceptable solutions is given graphically in figure 6.2

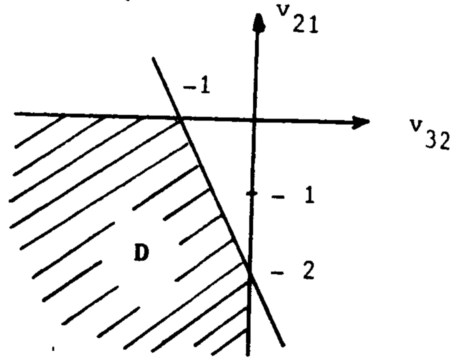

Fig. 6.2

Thus by selecting v_{21} and $v_{32} \in D$, the invariant system $\dot{Z}=(A+\bar{V})Z$ is asymptotically stable and according to theorem (6-4-2), the system $\dot{X}=[A+V(t)]X$ is on the average also stable. Note finally, that the amplitudes and frequencies of vibrations (q_{21} r_{21}, q_{32} & r_{32}) should be sufficiently high.

VI.4.3. Vibrational Feedback Laws

The dynamic matrices of the systems under consideration in this section will not be directly vibrationally stablizable. To stabilize this system class, we first will apply a (state) output feedback verifying the structural stress applied. This is to make the closed loop system vibrationally stabilizable. Then we will apply a vibrational control which stabilizes the closed loop system. Given the following system:

$$\dot{X}(t) = A\,X(t) + B\,U(t)$$
$$Y = C\,X(t) \qquad (6\text{-}4\text{-}13)$$

where A does not verify the conditions of theorem (6-4-1). Let us apply a control U = KY such that $K \in K_F$ (K_F: matrix set satisfying the structural stress applied), and such that the closed loop system:

$$\dot{X}(t) = (A + BKC)\,X(t) = D\,X(t) \qquad (6\text{-}4\text{-}14)$$

is vibrationally stabilizable. Then we stabilize system (6-4-14). If vibrations are introduced only on D elements dependent on the k_{ij} elements of K, then the vibrational control may be reduced to a time-variant (vibrational) feedback, verifying the structural stress. This idea is illustrated by the example below:

Example 6-4-2 (TRA-85a) /107

Given the following controllable and observable system with two stations:

$$\overset{\circ}{X} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u_2$$

$$y_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X$$

$$y_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} X$$

Given the following feedback matrix:

$$K = \left[\begin{array}{ccc:c} k_{11} & k_{12} & k_{13} & 0 \\ \hdashline 0 & 0 & 0 & k_{24} \\ 0 & 0 & 0 & k_{34} \end{array}\right]$$

which gives in a closed loop:

$$\overset{\circ}{X} = D\,X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_{11} & 1+k_{12} & 0 & k_{13} \\ 0 & k_{24} & 1+k_{24} & 0 \\ 0 & k_{34} & k_{34} & 1 \end{bmatrix} X \qquad (6\text{-}4\text{-}15)$$

It is easy to check whether the system has a decentralized fixed mode in s=1, and the output feedback can therefore not stabilize the system.

The open loop system is not vibrationally stabilizable (Tr LA = 3 >), but we can apply a vibrational control to the closed loop system (6-4-15) by selecting K such that:

$$Tr\ D = 3 + k_{12} + k_{24} < 0$$

Elements d_{21} and d_{42} of D are vibrationally controllable if $k_{13} \neq 0$ (for a lower quasi-triangular vibration. Let us therefore introduce the following vibrations to D:

$$V(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ q_{21} \sin r_{21} t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q_{42} \sin r_{42} t & 0 & 0 \end{bmatrix}$$

which gives the following matrix $\bar{V}$ (see remark 4 of section VI.4.1):

$$\bar{V} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\bar{v}_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -k_{13}\bar{v}_{42} & 0 & 0 \end{bmatrix} \qquad \bar{v}_{21} = \frac{q_{21}^2}{2\, r_{21}^2} > 0 \qquad \bar{v}_{42} = \frac{q_{42}^2}{2\, r_{42}^2} > 0$$ /108

The time-invariant system which describes on the average the variant system $\overset{\circ}{X}(t) = [D + V(t)]\ X$ is:

$$\overset{\circ}{Z} = (D + \bar{V})\ Z$$

$$\overset{\circ}{Z} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_{11} - \bar{v}_{21} & 1+k_{12} & 0 & k_{13} \\ 0 & k_{24} & 1+k_{24} & 0 \\ 0 & k_{34} - k_{13}\bar{v}_{42} & k_{34} & 1 \end{bmatrix} Z \qquad (6\text{-}4\text{-}16)$$

The typical polynomial of this system is:

$$\det(sI - D - V) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_o$$

with $a_3 = -(3 + k_{12} + k_{24})$

$$a_2 = 1+k_{12}+(1+k_{24})(2+k_{12}) + k_{13}(k_{13}\bar{v}_{42} - k_{34}) + (\bar{v}_{21} - k_{11})$$

$$a_1 = -(1+k_{24})\{(1+k_{12}) + k_{13}(k_{13}\bar{v}_{42} - k_{34})\} + (k_{11} - \bar{v}_{21})(2+k_{24}) - k_{13}k_{24}k_{34}$$

$$a_o = (1+k_{24})v_{21} - k_{11})$$

To introduce the desired response to the "average" system (6-4-6), we identify the coefficients of its typical polynomial with the coefficients of the desired polynomial, i.e.:

$$\det(sI-D-V) = (s+1)^4$$

which gives us: $a_3 = 4$, $a_2 = 6$, $a_1 = 4$ and $a_0 = 1$

we have four equations and seven unknowns, let us therefore set k_{24}, k_{13} and $\bar{v}_{21}$.

$$k_{24} = -3,\ k_{13} = 1 \text{ and } K_{11} = k_{11}-\bar{v}_{21}$$

By solving the set of equations we have:

$$k_{12} = -4,\ k_{34} = -\frac{1}{6},\ v_{42} = \frac{16}{3}\ \&\ K_{11} = k_{11}-\bar{v}_{21} = 0{,}5$$

for $\bar{v}_{21} = 1$ we obtain $k_{11} = 1.5$.

Remark: since the vibrationally controllable elements are elements of the feedback matrix K, the introduction of vibrations to d_{21} and d_{42} is equivalent to adding vibrations to the elements of matrix K and therefore the application of a time-variant (vibrational) output feedback of the following form:

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$$K(t)=\left[\begin{array}{cccc|c} 1{,}5 + q_{21}\sin r_{21}t & & -4 & 1 & 0 \\ \hline 0 & & 0 & 0 & -3 \\ 0 & & 0 & 0 & -\frac{1}{6} + q_{42}\sin r_{42}t \end{array}\right] \text{ with: } \quad \frac{q_{21}}{r_{21}} = \sqrt{2} \text{ and } \quad \frac{q_{42}}{r_{42}} = \sqrt{\frac{32}{3}}$$

Note again that the vibrations amplitudes and frequencies should sufficiently large (see Th. 6-4-2).

At this stage, the following question arises: For systems with unstable fixed modes does it suffice to have all vibrationally controllable elements of a closed loop system

(6-4-14) dependent on the elements of a feedback matrix K, to be able to stabilize the system by a decentralized vibrational feedback system? The answer is NO. Actually this condition is only necessary, and its inadequacy is illustrated by the following example:

Example 6-4-3

Given a controllable and observable two station system:

$$\mathring{X} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ \end{bmatrix} u_2$$

$$y_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} X$$

$$y_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

The system has a nonstructural decentralized fixed mode in s=1. The trace of matrix A is not negative. The system is therefore not vibrationally stabilizable in an open loop.

Let us apply a decentralized output feedback to the system K = diag (k_1, k_2). The closed loop system becomes:

$$\mathring{X} = DX = \begin{bmatrix} -1 & k_1 & k_1 \\ k_2 & 1 & k_2 \\ 0 & k_1 & 3+k_1 \end{bmatrix} X$$

this system is vibrationally controllable since the system verifies the conditions of Meerkov's theorem (6-4-1), for a proper selection of elements k_1 and k_2. The vibrationally controllable D elements are $d_{12} = k_1$ and $d_{23} = k_2$ (for a higher quasi-triangular structure). Although these elements depend on k_1 and k_2, stabilization using a vibrational feedback in the form:

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$$K = \left[\begin{array}{c|c} K_1(t) & 0 \\ \hline 0 & K_2(t) \end{array}\right] = \left[\begin{array}{c|c} k_1 + q_{12} \sin r_{12} t & 0 \\ \hline 0 & k_2 + q_{23} \sin r_{23} t \end{array}\right]$$

is not feasible since the following is obtained in a closed loop when this feedback is applied to the system:

$$\mathring{X}(t) = \left[A + B\,K(t)\,C\right] X = D(t)\,X = \begin{bmatrix} -1 & K_1(t) & K_1(t) \\ K_2(t) & 1 & K_2(t) \\ 0 & K_1(t) & 3+K_1(t) \end{bmatrix} X$$

Let us express D(t) + V(t), and the vibrations matrix is not quasi-triangular. Therefore theorem (6-4-2) is no longer applicable.

The following lemma proposes a necessary and sufficient condition for the existence of a vibrational feedback law that stabilizes the system in the presence of unstable nonstructural fixed modes.:

Lemma 7-4-1

Given system (C,A,B) with fixed modes with respect to K_F (K_F: set of acceptable control matrices). An acceptable vibrational feedback exists that stabilizes the system if:

1) the closed loop system X = (A+BKC) = DX, $K \in K_F$ is vibrationally stabilizable (see th. 6-4-1).

2) All vibrationally controllable elements of D depend on the elements of feedback matrix k, if and only if:

3) All k_{ij} elements contained in vibrationally controllable elements of D are placed on one side of the D diagonal.

4) At least one of the vibrational gains belongs to two D elements, one of them not being vibrationally controllable.

Demonstration

Condition 1) is direct according to theorem (6-4-1), and condition 2) is obvious because if one D vibrationally controllable element does not depend on $K \in K_F$, then an application of a vibrational feedback has no effect on this element. Let $k_{ij}(t)$ be one of the vibrational gains. If k_{ij} is contained in elements on both sides of the diagonal, in a closed loop, then the resulting vibrations matrix is not quasi-triangular, and theorem (6-4-2) is not applicable, which gives us condition 3). Let us assume that k_{ij} contained in a single vibrationally controllable D element is: $d_{s\ell} = Q_{s\ell} + f_{s\ell}\, k_{ij}$. If k_{ij} is replaced by $k_{ij}(t) = a \sin rt$, on the average $d_{s\ell} = a_{s\ell} + f_{s\ell}\,(k_{ij} + \bar{v})$ is obtained where $\bar{v}$ is a constant dependent on q, r and $d_{\ell s}$. Since the fixed modes are defined with respect to the structure of K and are independent of the K values, the change in k_i by $k_{ij}+v$ does not affect the fixed mode, which leads us to condition 4.

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Example 6-4-4

Let us take another look at example (6-4-2). We showed that the system can be stabilized by a vibrational output feedback in the form:

$$K(t)=\left[\begin{array}{ccc|c} k_{11}+a_{21}\sin r_{21}t & k_{12} & k_{13} & 0 \\ \hline 0 & 0 & 0 & k_{24} \\ 0 & 0 & 0 & k_{34}+q_{42}\sin r_{42}t \end{array}\right]$$

Since k_{11} belongs to a single D element (see 6-4-15), given lemma (7-4-1), the vibrations in k_{11} do not affect the stability of the system on the average. (The characteristic equation of system (6-4-16) can be calculated and verified for $\bar{v}_{42} = 0$, therefore $\forall\ \bar{v}_{21}$ the system has an unstable mode in s=1), and therefore by selecting $\bar{v}_{21} = 0$ (no vibrations in k_{11}), the vibrational feedback matrix becomes:

$$K(t)=\left[\begin{array}{ccc|c} 0,5 & -4 & 1 & 0 \\ \hline 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & \frac{-1}{6}+q_{42}\sin r_{42}t \end{array}\right] \text{ with } \frac{q_{42}}{r_{42}} = \sqrt{\frac{32}{3}}$$

Remark: the number of vibrational feedback loops has diminished.

The result of this section is similar to that of Anderson and Moore, presented in the preceding section. It shows that a time-variant feedback can stabilize unstable nonstructural fixed modes of the system. The advantage of a vibrational control, with respect to Anderson's and Moore's approach is that the control functions are continuous, and therefore easier to produce technologically. Moreover, feedback loops are not all time-variant.

VI.4.4. Stabilization By a Vibrational Control of systems With Structural Fixed Modes

It was established (KOB-82) that a system with structurally uncontrollable fixed modes (type i and ii_i structural fixed modes) is placed in a closed loop in the following block-triangular form: /112

$$\overset{\circ}{X} = A+BKC = DX = \left[\begin{array}{c|c|c|c} D_{11} & & & \\ \hline D_{21} & D_{22} & & 0 \\ \hline \vdots & & \ddots & \\ \hline D_{k1} & \cdots & \cdots & D_{kk} \end{array}\right] X \qquad (6\text{-}4\text{-}17)$$

where $X \in R^n$, $D_{ii} \in R^{n_i x n_i}$ & $\sum_{i=1}^{k} n_i = n$, where matrices D_{11} are irreducible matrices (i.e. in terms of graph theory: the states associated with block D_{ii} i=1,...,k are highly connected). The fixed modes of the system are block modes D_{ii} which do not depend on matrix K. Therefore, the stabilization of a system with structural fixed modes is the same as a system with a block-triangular structure of form (6-4-17).

According to the results of section VI.4.1., if a block-triangular is vibrationally stabilizable (verifies the conditions of Meerkov's theorem 6-4-1) then the vibrations should be introduced into vibrational controllable elements of diagonal blocks since, for the technological reasons already given, they cannot be introduced into zero elements. Consequently, the vibrations matrix should have a block-diagonal structure, with a structure of each lower (or higher) quasi-triangular block, since we do not know how to analyze the system with another structure (MEE-80). It is evident that it is not necessary to introduce vibrations in all blocks, but it is necessary and sufficient to introduce them into blocks with instabilities. In short, the vibrations matrix associated with the system with a block-triangular structure (6-4-17) should have the following form:

$$V(t) = \text{block diag.} \left[V_1(t), \ldots, V_k(t)\right] \quad V_i \in R^{n_i x n_i} \tag{6-4-18}$$

with:
$$V_i(t) = \begin{cases} \text{quasi-triangular if } D_{ii} \text{ has at least one stable mode} \\ 0 \quad \text{if } D_{ii} \text{ does not have an unstable mode} \end{cases}$$

Given that $V_i(t)$ has the same dimension as D_{ii}, and should have a quasi-triangular structure, it is evident, that the dimension of D_{ii}, which has instabilities, should be at least equal to two, in order for the vibrational control to be applicable to the system. The following corollary reformulates theorem (6-4-1) for the system class with a block-triangular structure of form (6-4-17):

Corollary 6-4-1

Given system $\mathring{X} = D X$ where D is a block-triangular matrix with irreducible diagonal blocks (6-4-17). On the assumption that a line vector c exists such that the pair (c,A) is observable, the

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system is vibrationally controllable if and only if the D trace is negative and the dimension of each D_{ii} block with instabilities is equal to at least two.

Now let us consider the algebraic characterization of Sezer's and Siljak's (SEZ-81b) structural fixed modes presented in section VI.4. Let us apply a decentralized control to system (4-4-1), we obtain the following in a close loop:

$$\overset{\circ}{X} = A+BKC = DX = \left[\begin{array}{c|c|c} D_{11}(K) & & \\ \cline{1-2} D_{21}(K) & D_{22} & \text{\huge 0} \\ \hline D_{31}(K) & D_{32}(K) & D_{33}(K) \end{array}\right] X \qquad (6\text{-}4\text{-}19)$$

Let us assume that the unstable modes of the system are all fixed modes (i.e. they belong to $D_{22} = A_{22}$ modes), as the unstable modes of blocks $D_{11}(K)$ and $D_{33}(K)$ may be stabilized by an appropriate choice of K. Given corollary 6-4-1, the stabilization of type i structural fixed modes of the system using a vibrational control is possible if and only if:

1 - c exists such that (c,A) is observable,
2 - Tr D < 0,
3 - When block D_{22} is put in an irreducible block-triangular form similar to (6-4-17), then the dimension of each sub-block with instabilities is equal to at least two.

It is evident that these conditions can be applied to an open loop system by replacing D by A.

<u>Note</u>:

1 - A system with only one type i unstable structural fixed mode cannot be stabilized by a vibrational control, unless the physical introduction of vibrations on zero elements is permissible.

2 - For systems with type i structural fixed modes, the vibrationally controllable elements (elements belonging to $D_{22} = A_{22}$ unchanged by the output feedback verifying the structural stress), never depend on elements of matrix K. We therefore conclude that a type i structural fixed mode can never be influenced by a time-variant feedback: such a conclusion is quite consistent with that of Sezer and Siljak (SEZ-81 a,b).

VI.4.5. Comments

The use of a vibrational control for stabilizing the unstable fixed modes of a system seem interesting because, owing to the fact it does not require system measurements, it adapts to the structural stress. In some cases, it may be also considered a time-variant feedback.

Despite these advantages, heavy* calculations are generally needed to determine an "average system" (integration of trigonometric functions). The same is true for determining vibrational amplitudes and frequencies (resolution of a nonlinear inequality set). Although this drawback is attenuated by the fact that these are off-line calculations and that they may be systemized for making software, we think that this control is not easily applicable for systems with a limited order.

VI.5 CONCLUSION

In this chapter we presented methods that currently exist in literature for stabilizing systems in the presence of nonstructural and unstable fixed modes. In contrast to the case of centralized and unstable fixed modes, we showed that systems with structurally controllable fixed modes under structural stress may be stabilized using unsteady (nonlinear, time-variant or vibrational) controls However, in sacrificing the property of invariance or linearity, we encounter analysis problems even for small-scale systems. Further, these controls re difficult to implement on a practical level. In the light of these difficulties, we concluded that the most natural approach for stabilizing the system or for placing its poles is to relief stresses on the control structure. Moreover, structural fixed modes (in some cases) may not be removed with this approach. This preserves the steadiness of the control, an aspect that will be studied in the next chapter.

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*except for a system class with only one vibrationally controllable element per line.

CHAPTER VII - POLE PLACEMENT OF DECENTRALIZED SYSTEMS: SELECTION OF THE CONTROL STRUCTURE /115

VII.1 INTRODUCTION

In the previous chapters, we have shown that fixed modes are divided into: fixed modes that are structurally controllable under structural stress (type ii_2 nonstructural and structural fixed modes, due to a perfect simplification between poles and zeros (see ch. IV) that an unsteady control can prevent (see ch. VI)) and modes that are structurally uncontrollable under structural stress (type i and ii structural fixed modes because information is not being transferred between the subsystems. This problem is fixed by adding extra links between the subystem, i.e. by relieving the stresses on the control structure.

Since the inclusion of a (time-variant or nonlinear) unsteady /116
or nonconventional (vibrational) control is difficult to do, it seems that structural stress relief (when possible) is the most natural approach for avoiding the occurrence of fixed modes of any type.

In this chapter, we shall present the different methods, currently existing in literature, that are used for determining an optimum control structure (with respect to the number of feedback loops or their related cost) without fixed modes; i.e. permitting free a free pole placement of the system. The methods of synthesizing the structural stress control of systems without have no fixed modes will be discussed in the next chapter.

If the system decomposition into subsystems is implemented (this is often the case, particularly for geographically distributed systems) then the problem is to find extra feedback loops to be added to the predefined control structure to remove the fixed modes. However if the system decomposition is not totally established, then the problem remains in finding the most decentralized control structure possible to allow for an arbitrary pole placement of the system.

VII.2 STRESS RELIEF ON THE CONTROL STRUCTURE

The existence of fixed modes is associated with the stress applied to the structure of the control matrix. The most natural method for removing them is then to relief this stress, namely to change the structure of the control matrix by adding extra links between the subystems. Let us consider system (C,A,B) of (2-4-2) and decentralized structural stresses (the set of block-diagonal control matrices K_d). If the system has fixed modes relative to K_D, the problem will be to find extra links to add between the subsystems so that the fixed modes disappear, i.e. to find a control structure K:

$$K = K_d + \{\text{certain} \quad K_{ij} / \ i \neq j\}$$

such that the fixed modes are eliminated (fig. 7.1).

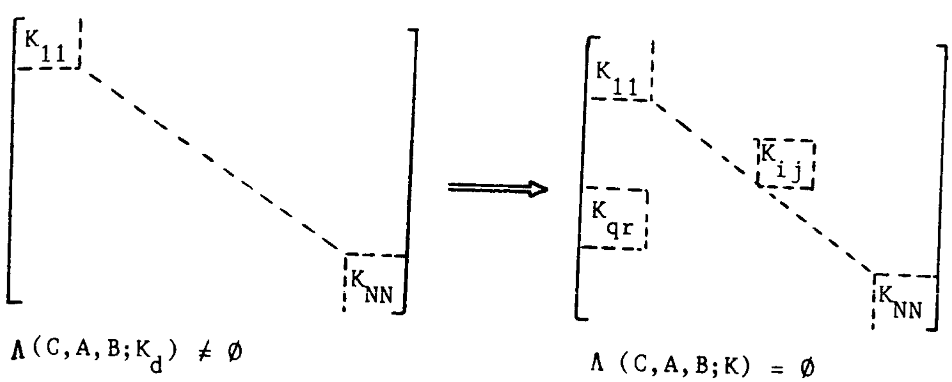

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Fig. 7.1

The determination of the new structure should be optimum, i.e. this new structure should satisfy an optimality criterion which is, for the methods presented in this section, either the number of extra links between subsystems, or their related cost.

VII.2.1. Wang's and Davisons' Approach

Wang and Davison (WAN-78a) searched for a control, for a system (C,A,B) with N stations, minimizing the cost of transferring information between stations and stabilizing the system. Let us consider the set of matrices K such that:

$$K(r_{ij},\ i,j=1,\dots,N) \triangleq \left\{ K \;/\; K = \mathrm{cost}(K_{ij}) = \begin{bmatrix} K_{11} & \cdots & K_{1N} \\ \vdots & & \vdots \\ K_{N1} & \cdots & K_{NN} \end{bmatrix}, \right.$$

$$\left. K_{ij} \in R^{m_i x p_i}, \mathrm{rank}\,(K_{ij}) = r_{ij} \right\} \qquad (7\text{-}2\text{-}1)$$

where r_{ij} is the number of links between station j and station i (corresponding to a local control $u_i = K_{ij}\ y_j$). If the cost of the link per unit of time from station i to station j is z_{ij}, then the total cost of the information transfer is:

$$\theta(r_{ij},\ i,j=1,\dots,N) = \sum_{j=1}^{N} \sum_{i=1}^{N} r_{ij}\ z_{ji}$$

The problem is therefore reduced to:

$$\underset{0 \leqslant r_{ij} \leqslant \min(m_i,p_j)}{\mathrm{Min}} \quad \theta(r_{ij},\ i,j=1,\dots,N)$$

$$\text{under} \quad \Lambda\left[C,A,B;K(r_{ij},\ i,j=1,\dots,N)\right] \subset \mathbf{C}^- \qquad (7\text{-}2\text{-}2)$$

where $\mathbf{C}^-$ is the left half of the complex plan.

For a given control structure, let us express each submatrix /118
K_{ij} of rank $_{ij}$ as the product of two matrices: $K_{ij} = L_{ij}$
M_{ij} where L_{ij} is of dimension $m_i \times r_{ij}$ and M_{ij}
is of dimension $r_{ij} \times p_j$. If $S_I = \{s_1,\dots,s_q\}$
is the set of unstable A modes, then condition (7-2-2) ils verified if and only if L_{ij} and M_{ij} exist such that:

$$\prod_{s_i \in S_I} \det\left[s_i I - A - B\left[\mathrm{bloc}(L_{ij}\ M_{ij})\right] C\right] \qquad (7\text{-}2\text{-}3)$$

is not identically zero.

Remark: Condition (7-2-3) implies that no S_I element is a fixed mode with respect to control $u = [\mathrm{block}\ (L_{ij}\ M_{ij})]Y$.

Since the possibilities for selecting integers r_{ij} are limited, the problem of optimization cannot be solved in a limited number of steps, as shown in the following example:

Example 7-2-1 (WAN-78a)

Let us consider the following two-station system (N=2)

$$\overset{\circ}{X} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u_2$$

$$y_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} X$$

$$y_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} X$$

Let us assume that the cost of transferring information between two stations is:

$z_{ii} = z_{22} = 0$, $a_{12} = 1$ and $z_{21} = 2$

1 - Let us consider the following decentralized control:

$$K_{11}(r_{11}=r_{22}=1,\ r_{12}=r_{21}=0) = \left\{ \left[\begin{array}{c|cc} k_{11} & 0 & 0 \\ k_{21} & 0 & 0 \\ \hline 0 & k_{32} & k_{33} \end{array}\right],\ k_{ij} \in R \right\}$$

The fixed polynomial with respect to K_i is:

$$\Psi(s;C,A,B,K_1) = \underset{k_{ij} \in R}{\text{p.g.c.d.}} \{ \det(sI-A-B\,K_1\,C) \} = s$$

2 - Let us consider the case of a nonzero transfer from station 1 to station 2, the cost, then, is: $\theta(r_{11}=r_{21}=r_{22}=1,\ r_{12}=0) = 1$

and the control matrix is: /119

$$K_2(r_{11}=r_{21}=r_{22}=1,\ r_{12}=0) = \left\{ \left[\begin{array}{c|cc} k_{11} & 0 & 0 \\ k_{21} & 0 & 0 \\ \hline k_{31} & k_{32} & k_{33} \end{array}\right],\ k_{ij} \in R \right\}$$

we can show that the fixed polynomial is always s.

3 - Now let us consider a transfer from station 2 to station 1. In this case the transfer cost is: $\theta(r_{11}=r_{12}=r_{22}=1,\ r_{21}=0) = 2$ and the

control matrix is:

$$K_3(r_{11}=r_{12}=r_{22}=1,\ r_{21}=0) = \left\{ \left[\begin{array}{c|cc} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ \hline 0 & k_{32} & k_{33} \end{array} \right],\ k_{ij} \in R,\ \text{rank} \begin{bmatrix} k_{12} & k_{13} \\ k_{22} & k_{23} \end{bmatrix} = 1 \right\}$$

The rank stress on matrix $\begin{bmatrix} k_{12} & k_{13} \\ k_{22} & k_{23} \end{bmatrix}$ may be expressed:

$$\begin{bmatrix} k_{12} & k_{13} \\ k_{22} & k_{23} \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix}$$

where $\delta_1, \delta_2, \gamma_1$ & γ_2 are arbitrary real numbers. The fixed polynomial of the system is:

$$\underset{\delta_i\ \gamma_i\ k_{ij}}{\text{p.g.c.d.}} \left\{ \det \begin{bmatrix} s-\delta_1\gamma_1 & -\delta_1\gamma_2 & -k_{11} \\ -\delta_2\gamma_1 & s-\delta_2\gamma_2 & -k_{21} \\ -k_{32} & -k_{33} & s+2 \end{bmatrix} \right\} = 1$$

and there are no more fixed modes if the following is selected:

$$\delta_1 = \delta_2 = k_{11} = 1,\quad \gamma_1 = k_{21} = k_{33} = 0 \quad \& \quad \gamma_2 = k_{32} = -1$$

and all poles of the system are placed at -1.

Wang's and Davison's procedure (WAN-78a) consists of calculating the cost of any structure enabling the system to be stabilized, then of selecting the structure with the lowest cost. Even though there are a limited number of steps, it seems that, for systems with a large number of stations, the approach is either difficult to apply, and therefore for real situations is not of great interest.

VII.2.2. Armentano's and Singh's Approach

Armento's and Singh's procedure (ARM-82) handles cases of interconnected systems and is based on the fixed mode characterization described in section (III.3.5b).

Let us consider system (C,A,B) of (3-3-8) and any structure /120
control K_F such that $K_F \in K_d$ where K_d is the set of block diagonal matrices, therefore: $\Lambda(C,A,B,K_F) \subset \Lambda(C,A,B,K_d)$. Assume that

s is a fixed mode of the system with respect to the two structures K_d and K_F, then by applying corollary (3-3-3) to $(A+B\ K_F)$ we see that there is at least one i $i \in \{1,\ldots,N\}$ such that:

$$\|(\hat{A}_{ii} - s\ I_i)^{-1}\|^{-1} \leqslant \sum_{\substack{j=1 \\ j\neq i}}^{N} \|A_{ii} + B_i\ K_{ij}\ C_j\| \qquad (7\text{-}2\text{-}4)$$

$$\forall\ K_{ii} \in R^{m_i x p_i},\ \forall\ K_{ij} \in R^{m_i x p_j}\ \&\ \ i \neq j$$

Since expression (7-2-4) is verified for any K_{ii} and K_{ij}, it is particularly verified for any $K_{ii} = 0$. Given the set of indices:

$S_d = \{i/i \in \{1,\ldots,N\}$ and (3-3-10) is verified$\}$

$S_F = \{i/i \in \{1,\ldots,N\}$ and (7-2-4) is verified$\}$

we therefore have:

$$S_F \subset S_d \qquad (7\text{-}2\text{-}5)$$

Proposition 7-2-1 (ARM-82)

If s is a fixed mode relative to two structures K_d and K_d, then s verifies (7-2-4) for indices $i \in S_F$ such that (7-2-5) is verified.

From proposition (7-2-5), we know that if s remains a fixed mode for another structure,then the K_{ij} may be set at zero for $i \in S_F$, and s verifies (7-2-4) for an index subset of S_d). This leads us to Armentano's and Singh's procedure: "to eliminate a decentralized fixed mode, simply add nondiagonal blocks $K_{ij} \neq 0$ to matrix K_d for indices $i \notin S_d$".
If the system has several fixed modes, it is simply necessary to consider the association of sufficient structures in eliminating each fixed mode.

Example 7-2-2 (ARM-82)

If we assume we have a system with three subsystems, as depicted on the next page:

$$
\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \hline \dot{x}_{21} \\ \dot{x}_{22} \\ \hline \dot{x}_{31} \\ \dot{x}_{32} \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_{11} \\ x_{12} \\ \hline x_{21} \\ x_{22} \\ \hline x_{31} \\ x_{32} \end{bmatrix} + \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
$$

by applying a decentralized control of the form:
we obtain the following in a closed loop: $u_i = \begin{bmatrix} k_{i_1} & k_{i_2} \end{bmatrix} \begin{bmatrix} x_{i_1} \\ x_{i_2} \end{bmatrix}$ $i=1,2,3$ /121

$$
A + BK = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ k_{11} & k_{12} & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & k_{21} & k_{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & k_{31} & k_{32} \end{array}\right]
$$

The system has a fixed mode at the origin s = 0.

Let us use the matrix norm $\|A\| = \max\limits_i \sum\limits_j |a_{ij}|$ to calculate (3-310) with $\hat{A}_{ii} = \begin{bmatrix} 0 & 1 \\ k_{i1} & k_{i2} \end{bmatrix}$ and s = 0, we obtain:

$$
\|(A_{ii})^{-1}\|^{-1} = \frac{1}{\max\left\{\left(\frac{|k_{i2}|}{|K_{i1}|} + \frac{1}{|k_{i1}|}\right),\ 1\right\}} \leq 1 \qquad i=2;3
$$

therefore s = 0 satisfies (3-3-10) for $S_d = \{2,3\}$. It is easily verified that in order for control structure K* s = 0 to remain a fixed mode, it is eliminated for K**.

$$
K^{**} = \begin{bmatrix} K_{11} & 0 & 0 \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \qquad K^{**} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix}
$$

Armentano's and Singh's procedure described above only determines the i indices of the extra blocks of the control matrix to be added in order to eliminate the fixed modes. It does not provide any information on the j indices or on the number of extra blocks required. In the case of the example, it is easily verified that if K_{13} is done in structure K**, the fixed mode remains eliminated. If the system has several fixed modes, the procedure can give several indices $i \notin S_d$ and therefore redundant extra links for eliminating fixed modes: the solution obtained is therefore far from optimum. Further, the procedure is usable only if S_d is a natural subset of set $\{1,2,\ldots,N\}$ as shown in the following example:

Example 7-2-3 (ARM-82)

Given system (C,A,B), with two subsystems:

$$\begin{bmatrix} \dot{x}_1 \\ \hline \dot{x}_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix} + \left[\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \begin{bmatrix} u_1 \\ \hline u_2 \end{bmatrix}$$

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$$\begin{bmatrix} y_1 \\ \hline y_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix}$$

For a decentralized control matrix K = block ($_{11}$, K_{22}) the closed loop matrix becomes:

$$A + BKC = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ k_{11} & k_{12} & 0 & 0 \\ \hline 1 & 0 & k_{21} & 1 \\ 0 & 0 & k_{22} & 0 \end{array}\right]$$

it is easy to verify whether the system has a fixed mode at the origin s = 0, we have:

$$\|(\hat{A}_{11})^{-1}\|^{-1} = \frac{1}{\max\left\{1, \left(\frac{1}{|k_{12}|} + \frac{|k_{11}|}{|k_{12}|}\right)\right\}} \leq 1$$

$$\|(\hat{A}_{22})^{-1}\|^{-1} = \frac{1}{\max\left\{\frac{1}{|k_{22}|} + \left(1 + \frac{|k_{21}|}{|k_{22}|}\right)\right\}} \leq 1$$

therefore S_d = {1,2} is not a natural subsystem of {1,2} and the procedure is not valid in this case!

VII.2.3. Approach Based on the Fixed Mode Sensitivity

The procedure proposed here is based on fixed mode characterization by their sensitivity (TAR-84b) described in section III.2.2.

VII.2.3a Formulation of the Problem

For a given structural stress, a fixed mode is not sensitive to control variations and therefore its gradient matrix SK_r is zero (see proposition 3-2-1). If structural stresses are not taken into consideration, then the nonzero elements of matrix SK_r correspond to the feedback matrix elements which may influence the mode under consideration. This leads us to the following proposition:

Proposition 7-2-2 (TAR-84b)

s_r is not a fixed mode of the system with respect to control $K \in K_F$ if matrix K contains at least one element of set K_{s_r} such that:

$$K_{s_r} = \{k_{ij}/ij \text{ such that } (sk_r)_{ij} \neq 0\}$$

where $(sk_r)_{ij}$ are elements of the sensitivity matrix with /123 respect to the control of mode s_r.

This proposition provides a simple method of determining the set of extra links used in eliminating the fixed modes from the system. Let us assume that the system has q fixed modes. According to proposition (7-2-2), we therefore have q sets of k_{ij} elements, i.e. K_{s_q}, associated with the fixed modes. (The elements of sets K_{s_i} i=1,...,q are determined from matrices SK_i which are calculated without considering the structural stress). The set of extra links K* searched for should therefore verify:

$$\text{Card}\,(K^* \cap K_{s_i}) \geq 1 \qquad i=1,\dots,q$$

Let us note that to determine the structure of K*, it is not necessary to consider all K_{s_i} sets for each mode:

$$\text{If} \quad K_{s_i} \subsetneq K_{s_j}, \quad i \neq j, \qquad \text{card}\,(K^* \cap k_{s_i}) \geq 1$$

$\Longrightarrow$ $\text{card}\,(K^* \cap K_{s_j}) \geq 1$ then K_{s_j} can be eliminated.

The problem is reduced to the following:

Problem 7-2-1

Find K* such that:

$$\text{card}\,(K^* \cap K_{s_i}) \geq 1 \qquad i=1,\dots,\bar{q} \leq q$$
$$K_{s_i} \not\subseteq K_{s_j}, \quad i \neq j$$

This solution to this problem may be found by solving the following optimization problem:

VII.2.3b Formulation of the Optimization Problem (TAR-84b and 85c)

Let us consider the set of elements formed by associating all K_{s_i} elements retained:

$$Z = \bigcup_{i=1}^{\bar{q}} K_{s_i} \qquad K_{s_i} \not\subseteq K_{s_j} \quad s_i \neq s_j$$

$$\text{Card } Z = r \leq m \times p$$

It is clear that each Z element represents a feedback link (for the sake of convenience, the preceding notation k_{ij} will be replaced by a_i i=1,...,r, for Z elements).

Let us associate a cost $r_i \geq 0$ for each Z feedback loop z_i and let us define the Boolean vector:

$$W = (w_1, \ldots, w_r)^T$$

with $$w_i = \begin{cases} 1 & \text{if } z_i \in K^* \\ 0 & \text{otherwise} \end{cases}$$ /124

Let us also define the following matrix:

$$L = \| \ell_{ij} \|_{\substack{i=1,\ldots,\bar{q} \\ j=1,\ldots,r}} \quad \text{with} \quad \ell_{ij} = \begin{cases} 1 & \text{if } z_j \in K_{s_i} \\ 0 & \text{otherwise} \end{cases}$$

The determination of the structure of matrix K (i.e. K*), which minimizes the cost of links and avoids fixed modes (see Prob. 7-2-1) may be formulated according to the following Boolean linear program:

Problem 7-2-2

$$\min \sum_{j=1}^{r} r_j\, w_j$$

$$\text{sub} \quad \sum_{j=1}^{r} \ell_{ij}\, w_j \geq 1 \qquad i=1,\ldots,\bar{q}$$

$$w_j = \begin{cases} 1 \\ 0 \end{cases} \qquad \ell_{ij} = \begin{cases} 1 \\ 0 \end{cases} \qquad \begin{array}{l} i=1,\ldots,r \\ j=1,\ldots,\bar{q} \end{array}$$

We recognize here the well-known graph theory problem in searching for a coverage of K by Z at minimum cost:

Given a unidirectional graph $G = [Z, K, \Delta]$ where:

$Z = \{z_1, \ldots, z_r\}$

$K = \{k_1, \ldots, k_{\bar{q}}\}$: set of Z parts

Δ : specific application of A in Z

$\Delta(z_i) = \{k_j \,/\, z_i \in k_j\}$ $i=1,\ldots,r$

A cost r_i is associated with each peak z_i $i=11,\ldots,r$.

An equivalent problem to problem (7-2-2) is:

Problem 7-2-3

$$\text{Find} \quad \xi \subseteq Z \,/\, \bigcup_{z_i \in \xi} \Delta(z_i) = K$$

$$\text{by minimizing} \sum_{z_i \in \xi} r_i$$

Several algorithms are at our disposal for solving this problem (7-2-3): Coverage method (KAU-68) (ROY-70), Gomorey's method (KAU-68), and Thiriez's method (THI-71).

Example 7-2-4 /125

Let us reexamine example (4-5-1). The system with two fixed modes (one is structural and the other not): $\Lambda = \{1,2\}$ the corresponding sensitivity matrices, without structural stress are:

$$SK(s) = \begin{bmatrix} 0 & 0 & -1/3 \\ 0 & 0 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}, \quad SK(s=2) = \begin{bmatrix} 0 & 0 & 0 \\ 1/3 & 0 & 1/6 \\ 0 & 0 & 0 \end{bmatrix}$$

We therefore have:

$$K_{s_1} = \{ k_{13}, k_{23} \}$$
$$K_{s_2} = \{ k_{21}, k_{23} \}$$

Set Z is:

$$Z = \{k_{13}, k_{23}, k_{21}\} = \{z_1, z_2, z_3\}$$

Matrix L is expressed:

$$L = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The linear program takes on the following form:

1 - If the link costs are equal $r_i = 1 \quad i=1,2,3$ (minimization of number of links) we then have:

$$\min (w_1 + w_2 + w_3)$$
$$\begin{aligned} w_1 + w_2 &\geq 1 \\ w_2 + w_3 &\geq 1 \end{aligned} \qquad w_i = \begin{cases} 1 \\ 0 \end{cases} \qquad i=1,2,3$$

The program solution is W = (0 1 0), and the addition of k_{23} is sufficient for eliminating the two fixed modes.

2 - However if we assume that the link costs are given by vector R = (1 3 2) then we have:

$$\min (w_1 + 3\, w_2 + 2\, w_3)$$
$$\text{sub } \begin{aligned} w_1 \; w_2 &\geq 1 \\ w_2 + w_3 &\geq 1 \end{aligned} \qquad w_i = \begin{cases} 1 \\ 0 \end{cases} \qquad i=1,2,3$$

The program gives two solutions:

$W = (1\ 0\ 1)$ which corresponds to elements k_{13} and k_{21}
$W = (0\ 1\ 0)$ which corresponds to element k_{23}.

VII.2.4. Characterization of a Set Containing Enough Links to Eliminate Type i Structural Fixed Modes

In this section, we are interested in systems containing type i structural fixed modes only, by characterizing the set containing the required number of extra feedback loops to be added in order to eliminate the fixed modes. To do this we shall use the algebraic characterizations of structural fixed modes (see ch. IV).1

VII.2.4a use of Sezer's and Siljak's Characterization

Sezer & Siljak (SEZ-81b) have shown that the state equation of a system with type i structural fixed modes may be put in the following form:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \left[\begin{array}{c|c|c} A_{11} & 0 & 0 \\ \hline A_{21} & A_{22} & 0 \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \left[\begin{array}{c|c} B_\alpha^1 & 0 \\ \hline B_\alpha^2 & 0 \\ \hline B_\alpha^3 & 0_\beta \end{array}\right] \begin{bmatrix} U_\alpha \\ U_\beta \end{bmatrix} \tag{7-2-6}$$

$$\begin{bmatrix} Y_\alpha \\ Y_\beta \end{bmatrix} \left[\begin{array}{c|c|c} C_\alpha^1 & 0 & 0 \\ \hline C_\beta^1 & C_\beta^2 & C_\beta^3 \end{array}\right] X$$

where the control and observation stations are partitioned into two aggregated stations α and β.

It is clear that the fixed modes relative to the control:

$$K = \text{bloc diag. } (K_\alpha, K_\beta) \tag{7-2-7}$$

are modes of block A_{22}. These modes (see ch. IV) are simultaneously uncontrollable by one of the aggregated stations, i.e. β, and are unobservable from the other, i.e. α. Since the system is assumed to be globally controllable and observable, then the fixed modes of the system are controllable from the aggregated station α and are observable from aggregated station β. Therefore, the set of extra and sufficient links for eliminating these modes is given by:

$$K_{\alpha\beta}=\{k_{ij} \ / \ ij \text{ so that } u_i \in U_\alpha \ \& \ y_{ij} \in Y_\beta\} \tag{7-2-8}$$

and therefore the feedback matrix becomes:

$$K' = \left[\begin{array}{c|c} K_\alpha & K_{\alpha\beta} \\ \hline 0 & K_\beta \end{array}\right] \tag{7-2-9}$$

Since the decomposition of the system into subsystems is done relative to the control and observation systems and not relative to the states, in; some cases the sufficient set $K_{\alpha\beta}$ may be reduced as follows:

Proposition 7-2-3

Since system (7-2-6) has type i structural fixed modes, the set of extra and sufficient feedback links $K_{suf} = \{k_{ij} \ / \ i \in I \ \& \ j \in J\}$. where I(J) is the set of unidentically zero column (line) indices of matrix B* (C*), with: /127

$$B^* = \begin{bmatrix} B_\alpha^1 \\ B_\alpha^2 \end{bmatrix} \quad \text{and } C^* = \begin{bmatrix} C_\beta^2 & C_\beta^3 \end{bmatrix}$$

Demonstration

The sufficiency of set K_{suf} is demonstrated simply and directly by calculating the dynamic matrix of a closed loop system and by showing that it is not block-triangular.

By applying control U = U Y (where K is in form 7-2-7) to system (7-2-6), the dynamic closed loop matrix is: D = A + BKC

$$D = \left[\begin{array}{c|c|c} B_\alpha^1 K_\alpha C_\alpha^1 + A_{11} & 0 & 0 \\ \hline B_\alpha^2 K_\alpha C_\alpha^1 + A_{32} & A_{22} & 0 \\ \hline B_\alpha^3 K_\alpha C_\alpha^1 + B_\beta^3 K_\beta C_\beta^1 + A_{31} & B_\beta^3 K_\beta C_\beta^2 & B_\beta^3 K_\beta C_\beta^3 \end{array}\right] = \left[\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right] \tag{7-2-10}$$

Thus, matrix D has the same block-triangular structure as A, and the structural fixed modes $\sigma(A_{22})$ remain unchanged.

It is clear that to eliminate these fixed modes, a feedback matrix is necessary to destroy the block-triangular D-structure and to therefore influence block D_{12}.

Now let us consider control structure (7-2-9), the closed loop matrix then becomes:

$$D' = D + \left[\begin{array}{c|c:c} B_\alpha^1 K_{\alpha\beta} C_\beta^1 & B_\alpha^1 K_{\alpha\beta} C_\beta^2 & B_\alpha^1 K_{\alpha\beta} C_\beta^3 \\ B_\alpha^2 K_{\alpha\beta} C_\beta^1 & B_\alpha^2 K_{\alpha\beta} C_\beta^2 & B_\alpha^2 K_{\alpha\beta} C_\beta^3 \\ \hline B_\alpha^3 K_{\alpha\beta} C_\beta^1 & B_\alpha^3 K_{\alpha\beta} C_\beta^2 & B_\alpha^3 K_{\alpha\beta} C_\beta^3 \end{array}\right] = \left[\begin{array}{c|c} D'_{11} & D'_{12} \\ \hline D'_{21} & D'_{22} \end{array}\right]$$

with

$$D'_{12} = \left[\begin{array}{c:c} B_\alpha^1 K_{\alpha\beta} C_\beta^2 & B_\alpha^1 K_{\alpha\beta} C_\beta^3 \\ \hdashline B_\alpha^2 K_{\alpha\beta} C_\beta^2 + A_{22} & B_\alpha^2 K_{\alpha\beta} C_\beta^3 \end{array}\right]$$

The block-triangular structure is destroyed and $K_{\alpha\beta}$ is the set of links we are looking for.

To find K_{suf} note that D'_{12} is expressed:

$$D'_{12} = \begin{bmatrix} B_\alpha^1 \\ B_\alpha^2 \end{bmatrix} K_{\alpha\beta} \begin{bmatrix} C_\beta^2 & C_\beta^3 \end{bmatrix} = D_{12} + B^* K_{\alpha\beta} C^*$$

$$D'_{12} = D_{12} + \sum_{i,j} (b^i)^* k_{ij} (C_j)^* \qquad (7\text{-}2\text{-}11)$$

where $(b^i)^*$ is the i-th column of B^* and $(Cj)^*$ is the j-th line of C^* and $k_{ij} \in K_{\alpha\beta}$. Formula (7-2-11) allows us to conclude that if $(b^i)^*$ or $(C_j)^*$ is identically zero, then k_{ij} has /128
no influence and there is no need to consider it.

Furthermore, the set K_{suf} is not empty since the system is controllable and observable which implies $B_\alpha^1 \neq 0$ & $B_\beta^3 \neq 0$. We stress the fact that K_{suf} is the set of extra K_{ij} elements to be added to the initial structure (7-2-7), that is assumed to be given, but if the structure is not given then we use the same method described above to show that the structure:

$$K'' = \left[\begin{array}{c|c} 0 & K_{\alpha\beta} \\ \hline K_{\beta\alpha} & 0 \end{array}\right] \qquad (7\text{-}2\text{-}12)$$

makes it possible to avoid type i structural fixed modes. For this structure, $K_{\alpha\beta}$ is given by proposition (7-2-3) and $K_{\beta\alpha}$ is given by:

$$K_{\beta\alpha} = \{ k_{ij} \ / \ ij \text{ so that } u_i \in U_\beta \ \& \ y_j \in Y_\alpha \} \qquad (7\text{-}2\text{-}13)$$

$K_{\beta\alpha}$ may be zero in the particular case where $U_\alpha(Y_\beta)$ controls (observes) the entire state space.

Example 7-2-5)

Given example (3-3-1). By the following permutation matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix}$$

The system is put in the following form:

$$P^T A P = \left[\begin{array}{ccc|c|c} 0 & 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ 1 & 0 & 0 & & \\ \hline 0 & 0 & a & & \\ \hline 0 & 0 & 1 & b & \\ \hline 0 & 0 & 0 & 0 & 3 \end{array}\right], \quad P B^T = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \underbrace{\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]}_{\alpha \qquad \beta} \end{array}$$

$$CP = \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \left[\begin{array}{ccc|c|c} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\end{array}}\right\} \alpha \\ \left.\vphantom{0}\right\} \beta \end{array}$$

and the system has a type i structural fixed mode relative to the control structure:

$$K = \left[\begin{array}{cc|c} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ \hline 0 & 0 & k_{33} \end{array}\right]$$

We therefore obtain $I = \{1,2\}$ and $J = \{3\}$ which gives $K_{suf} = \{k_{13}, k_{23}\}$ therefore: /129

$$K' = K + K_{suf} = \left[\begin{array}{cc|c} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ \hline 0 & 0 & k_{33} \end{array}\right] \quad \text{and} \quad K'' = \left[\begin{array}{cc|c} 0 & 0 & k_{13} \\ 0 & 0 & k_{23} \\ \hline k_{31} & k_{32} & 0 \end{array}\right]$$

Note that K' and K'' assure the absence of type i structural fixed modes and that the choice between the two is made as a function of an additional criterion (number of links or their cost, for example).

VII.2.4b Use of Sensitivity Characterization

According to Sezer's and Silajk's characterization (eq. 7-2-6), it is very simple to determine set K_{suf}, if the system is already in block-triangular form (7-2-6), but if the state equation of the system is in a general form, then the problem is to find a permutation matrix that puts it in this form.

This problem does not arise if we use structural fixed modes characterization based on their sensitivity (TAR-84b) (see IV.5). In effect, the structural sensitivity matrix (see def. 4-5-1) makes it possible to determine the elements of dynamic matrix A which are at the origin of type i structural fixed modes. Namely, the elements of block A_{22} (see eq. 7-2-6) and therefore the X_2 states that correspond to fixed modes. Knowing that the X_2 are uncontrollable from aggregated station β and unobservable from aggregated station α, using the accessibility matrix of system R, which is in the following form (see appendix 3 for its determination):

$$\begin{array}{cc} & \begin{array}{ccc} X & U & Y \end{array} \\ R = & \begin{bmatrix} E & F & 0 \\ 0 & 0 & 0 \\ Q & H & 0 \end{bmatrix} \end{array} \begin{array}{c} X \\ U \\ Y \end{array} \qquad (7\text{-}2\text{-}14)$$

we determine two stations α and β and therefore the K_{suf} set we are looking for. To to this, we propose the following algorithm:

Algorithm 7-2-1 (TAR-84b)

1 - Let Λ be the set of type i structural fixed modes of system (C,A,B) under consideration

2 - Determine the structural sensitivity matrix corresponding to the set of fixed modes Λ as follows:

$$SS = SS_1 + \ldots + SS_i \ F + \ldots + SS_r$$

where SS_i is the structural sensitivity matrix corresponding to the mode $s_i \in \Lambda$ and where "+" is the "OR logic".

3 - Determine the set of X_2 states corresponding to the fixed /130
modes Λ: $s_i \in X_2$ if at least one nonzero element exists in line or column i of matrix SS.

4 - Determine the accessibility matrix R of the graph associated with the open loop system (C,A,B) (see appendix 3).

5 - The X_2 states are uncontrollable from aggregated station β, therefore:

$u_j \in U_\beta$ if for $x_i \in X_2$ we have $f_{ij} = 0$

$\rightarrow U_\alpha = U - U_\beta$.

6 - The X_2 states are unobservable from aggregated station α, therefore:

$y_j \in Y_\alpha$ if for $x_i \in X_2$ we have $q_{ji} = 0$.

$\rightarrow Y_\beta = Y - Y_\alpha$.

7 - The set of supplementary and sufficient links for eliminating type i structural fixed modes is:

$K'_{suf} = \{k_{ij} \ / \ ij$ so that $u_i \in U_\alpha$ & $y_j \in Y_\beta \}$

Example 7-2-6

Given example (3-3-1) with $a \neq c$, we have seen that this system has a type i structural fixed mode, i.e. s= b.

1 - The structural sensitivity matrix corresponding s = b is:

$$SS(s=b)=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

2 - $X_2 = \{X_3\}$

3 - the lines (columns) corresponding to state X_3 of the accessibility matrix for (output) controls are:

$$\begin{array}{cc} & \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ x_3 & \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{array} \longleftarrow \text{3rd line of } F$$

$$\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

3rd line of Q

4 - The two aggregated stations are:

$$U_\beta = \{u_1, u_3\} \longrightarrow U_\alpha = \{u_2\}$$
$$Y_\alpha = \{y_1, y_2\} \longrightarrow Y_\beta = \{y_3\}$$

5 - The sufficient set is $K'_{suf} = \{k_{23}\}$ and the sufficient structure for eliminating fixed modes is: /131

$$\begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & k_{23} \\ 0 & 0 & k_{33} \end{bmatrix}$$

Note that set K'_{suf} is not equal to K_{suf}, determined in the preceding section, because in stage 4 of the algorithm proposed, stations α and β are determined from the accessibility matrix of the open loop system (C,A,B). Therefore the diagrams created by the K feedback of (7-2-7) are not taken into consideration. To accomplish this we obtain $K'_{suf} \subseteq K_{suf}$. However, K_{suf} of the algorithm proposed may be determined by replacing in step 4 of the algorithm the accessibility matrix of the open loop system (C,A,B) by that of the closed loop system (C,D,B) where D is the dynamic matrix of the closed loop system given in (7-2-10) and for an arbitrary K value.

It is evident that once stations and β are determined, there is no problem in determining K' or K".

Let us note that in general the set K_{suf} (or $K_{suf'}$), and therefore K' or K", is only formally of interest because it is simply <u>sufficient</u> and nothing allows us to specify the number of k_{ij} elements <u>needed</u> for eliminating fixed modes or for determining them. All we know is that the use of all K_{suf} (or K'_{suf}) elements assures this elimination.

VII.2.5. Comments

The approaches presented in this section are all based on the idea that by relieving the structural stress, the fixed modes can be eliminated. This approach seems to be the most natural, particularly particularly for eliminating structural fixed modes for which it is the only possible approach.

Wang's and Davisons' approach (WAN-78a) is impractical for large systems, and that of Armentano and Singh (ARM-82) gives only a rough solution to the problem (they only indicate the i index of the blocks to be added for eliminating the fixed modes), and this is only valid for the class of interconnected systems. The approach based on the sensitivity of fixed modes (TAR-84b) is valid for system with simple fixed modes. Its advantage is that it is simple to use. The approaches of section (VII.2.4) are for systems with type i structural fixed modes is limited and they are impractical.

We would like to emphasize a special case of structural stress /132
stress relief: this is the use of a global control, i.e. $\upsilon = u_\ell + u_g$ where u_ℓ is a local control which can be calculated in a traditional manner (CHE-70, SIL-69, THA-60) and u_g a global [static (SIL-76a and b, SUN-77, SIL-78, SOL-81 and 85) or dynamic (GRU-80, GRU-84)] control, developed from the states of all subsystems. This

approach does not account for the information transfer limitations caused by structural stress. Consequently, this approach is a centralized approach to the problem.

VII.3 POLE PLACEMENT BY DYNAMIC OUTPUT FEEDBACK: SELECTING THE CONTROL STRUCTURE

Although the decomposition of a system into subsystems is generally determined by physical considerations (control station distribution for example), it may have applications where the decomposition is not complete, i.e. a few degrees of freedom are available for selecting a decomposition. In this case, the problem is to find a control structure assuring free pole placement of the system (absence of fixed modes for a dynamic control). Secondary criteria may be added such as the number of feedback links or their related cost.

It is evident that a control structure can be determined using any characterization method (see ch. III, IV and V) and by testing all possible decompositions and by selecting that which verifies the set criterion. However this test is very heavy and requires many calculations as we have $2^{mp}-1$ configurations to test for a system with m inputs and p outputs. This is why it is advantageous to develop a decomposition method.

Below, we shall describe in detail procedures that currently exist in literature and used for determining an optimum control structure (relative to the number of feedback loops or their related cost) assuring the absence of fixed modes.

VII.3.1. Locatelli's, Schiavoni's and Tarantini's Approach

Locatelli, Schiavioni and Tarantini (LOC-77) use their graphic characterization of fixed modes (see section V.2) to search for a minimum set of feedback loops $F^* \subset F$ (F defined in section V.2) such that the mode set $\{s_1,...,s_h\} \subset \sigma(A)$ is assignable, i.e. the element of the set $\{s_1,...,s_h\}$ should be a fixed mode relative to F*). The minimum set F* is determined relative to a criterion accounting for costs associated with feedback loops: /133

$$\mathbf{R}(F) = \sum_{(i,j)\in F} r_{ij}$$

where r_{ij} is the cost associated with the authorized feedback loop k_{ij}.

This problem has a solution if and only if the system does not have fixed modes relative to set F. This solution is given by solving the following Boolean program:

$$\text{Min} \sum_{(i,j)\in E_{2F}} r_{i-m,j}\, w_{ij}$$

under stresses:

$$\left.\begin{array}{ll} \text{a} & \displaystyle\sum_{(i,j)\in E_{1F}} v^{g}_{ij}\, z^{g}_{ij} \geq 1 \\ \text{b} & \displaystyle\sum_{j/(i,j)\in E_F} z^{g}_{ij} = \sum_{j/(j,i)\in E_F} z^{g}_{ji} \qquad i \in V_F \\ \text{c} & z^{g}_{ij} \leq w^{g}_{ij} \qquad (i,j) \in E_{2F} \end{array}\right\} \quad g=1,\ldots,h$$

with $$w_{ij} = \begin{cases} 1 & \text{if } (i-m,j) \in F^{*} \\ 0 & \text{if } (i-m,j) \notin F^{*} \end{cases} \qquad (i,j) \in F_{2F}$$

and for $g = 1,\ldots,h$

$$v^{g}_{ij} = \begin{cases} 0 & (i,j) \in E_{1F},\ G_{j-m,i}(s_g) \neq 0, \infty \quad (s_g \text{ is neither a pole nor zero}) \\ 1 & (i,j)\ \ E_{1F},\ G_{j-m,i}(s_g) = \infty \quad (s_g \text{ is a pole}) \\ -q & (i,j)\ \ E_{1F},\ \displaystyle\lim_{s\to s_g} \frac{G_{j-m,i}(s_g)}{(s-s_g)^q} \neq 0, \infty \quad (s_g \text{ is a zero of order } q) \end{cases}$$

$$z^{g}_{i,j} = \begin{cases} 1 & \text{if the arc } (i,j) \text{ belongs to the elementary circuit with } s_g \text{ as a pole and such that the arcs } (i,j) \in E_{2F} \text{ included in this circuit minimize} \\ 0 & \text{otherwise} \end{cases}$$

Note that stress a guarantees that s_g is a pole of a transmittance from at least one arc retained, and stress b forces this arc to belong to an elementary program.

This program is very interesting because with slight modifications, it can be used to solve diverse problems (LOC-77) among which:

1 - Test for the existence of fixed modes,

2 - Search for a minimum structure, relative to the number of links between subsystems, to avoid fixed modes,

3 - Search for supplementary links to be added to an initial structure F, for which the system has fixed modes, used for eliminating fixed modes, while minimizing the cost of these links.

To accomplish this, simply replace set F by set P X M (where P = {1,...,p} and M ={1,...,m} in the initial problem and associated a very high cost to the elements of set (P X M - F) compared to the cost associated with the elements of set F.

Example 7-3-1

Given example (5-2-1), the system has a fixed mode s = b relative to the decentralized structure F_d {(1,1,(2,2),(3,3)}. To determine a minimum structure eliminating the fixed modes, we use Locatelleli et al's program to:

$$\Lambda = \{ b \} , \; F = P \times M \;\; \text{et}$$

$$r_{ij} = \begin{cases} 0 & (i,j) \in F_d \\ 1 & (i,j) \in F - F_d \end{cases}$$

we obtain :

$$\text{Min} \;\; w_{42} + w_{43} + w_{51} + w_{53} + w_{61} + w_{62}$$

under

ⓐ $$z_{26} \geqslant 1$$

ⓑ
$$z_{15} = z_{41} + z_{51} + z_{61} , \quad z_{41} + z_{42} + z_{43} = z_{24}$$
$$z_{24}+z_{26} = z_{42} + z_{52} + z_{62} , \quad z_{51} + z_{52} + z_{53} = z_{15}$$
$$z_{36} = z_{43} + z_{53} + z_{63} , \quad z_{61} + z_{62} + z_{63} = z_{26} + z_{36}$$

© $z_{ij} \leqslant w_{ij}$ $i=4,5,6$ & $j=1,2,3$

The program gives only one solution which corresponds to the feedback loop k_{23} and therefore the structure

$$\begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & k_{23} \\ 0 & 0 & k_{33} \end{bmatrix}$$

eliminates the fixed mode.

Note that Locatelli et al's program is only valid if the system has simple fixed modes, and that it does not give all possible solutions to the problem. In effect, in this example the structure:

$$\begin{bmatrix} k_{11} & 0 & k_{13} \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix}$$

is an optimum solution to the problem which is not obtained by the program. This leads to the question: What solutions are obtained by this program?

VII.3.2 Senning's Approach /135

Given system (2-4-2), Senning (SEN-79) looked for a feasiibly decentralization control defined as follows

Definition 7-3-1 (SEN-79)

A control structure is said to be feasibly decentralized if the system is stabilizable by this control structure and if the information transfer cost is minimal.

Senning examined the conventional control problem (based on a quadratic optimization criterion) of linear systems and searched for an optimum control structure relative to a criterion taking into account the system decomposition and the cost of links between the different subsystems. The solution to this problem gives a feasibly decentralized control in the form:

$$u_i = K_{ii}\, y_i + \sum_{j=1}^{N} K_{ij}\, y_j \qquad i=1,\ldots,N \tag{7-3-1}$$

Accordingly, Senning defined an expanded optimization criterion (EOC):

$$EOC = PI + \sum_{i=1}^{N} q_i^2 \tag{7-3-2}$$

which is the sum of the Performance Index (PI) and a term describing the desired control structure, using a weighted measurement of nonlocal information. This measurement is made in the form of a nonlocal control vector standard weighted by r_{ij} factors more or less penalizing the information exchange between two subsystems:

$$\begin{aligned} q_i &= \| U_{i,\ \text{non local}} \| \\ &= \Big\| \sum_{\substack{j=1 \\ i \neq j}}^{N} r_{ij}\, u_i \Big\| = \Big\| \sum_{\substack{j=1 \\ j \neq i}}^{N} r_{ij}\, K_{ij}\, y_j \Big\| = \| K_i\, R_i\, Y \| \end{aligned}$$

with

$$R_i = \begin{bmatrix} r_{ij}\, I_{p_1} & & & & & \\ & \ddots & & & 0 & \\ & & r_{i,i-1}\, I_{p_{i-1}} & & & \\ & & & 0 & & \\ & 0 & & & r_{i,i+1}\, I_{p_{i+1}} & \\ & & & & & \ddots \\ & & & & & & r_{iN}\, I_{p_N} \end{bmatrix}$$ /136

$$K_i = \left[K_{i,1}, \ldots, K_{i,i-1}, 0, K_{i,i+1}, \ldots, K_N \right]$$

where I_{p_i} is the identity matrix of dimension $p_i \times p_i$

The EOC criterion is expressed:

$$EOC = PI + \sum_{i=1}^{N} \| K_i \; R_i \; Y \|^2$$

and the problem posed by Senning is therefore: to find optimum control matrices $K_1^*, \ldots, K_N^*$ such that: $EOC\ (K_1^*, \ldots, K_N^*) \leq EOC\ (K_1, \ldots, K_N)$ y for any feasible matrix $K_1, \ldots, K_N$.

The solution to this optimization problem is given by Senning (SEN-79) p. 55). It can be obtained by solving four matricial problems for each station.

Senning's procedure is highly interesting, since it simultaneously handles the problem of a control structure and the conventional problem of system quadratic optimization. It therefore makes it possible to obtain optimum controls from the standpoint of structure and gain, without having to first test for the existence of fixed modes. Of course, if the system does not have fixed modes, then the control obtained is totally decentralized.

VIII.3.3. Procedures Based on the Graphic Characterization of V.4

The procedures presented here are based on the graphic characterization of Pichai et al's (PIC-84) and Linnemann's (LIN-83) structural fixed modes presented in section V.4.

VII.3.3a. Structure With Minimum Number of Elements

Let us consider system (C,A,B) and its directed graph described in section VII.2.4. Since the system is controllable and observable, then each state peak $x_k \in X$ k=1,...,n belongs to at least one input-output path $u_i \rightarrow y_j$, therefore according to theorem (5-4-1), it is possible to obtain a set of feedback loops that verify condition i of theorem (5-4-1), by considering the feedback loops corresponding to the arcs that close all input-output paths. Namely, it is a sufficient set for eliminating type structural fixed modes. This set is therefore given by:

$K_i = \{k_{ij} / ij \text{ such that } h_{ij} = 1\}$

where h_{ij} are elements of the input-output accessibility matrix H (see appendix 3).

Of course, this set is not minimum for some state peaks belonging to several input-output paths. This set will then be reduced afterward.

The binary vector Z_{ij} of dimension n, given by the following is associated with element k_{ij}: /137

$$z_{ij}^t = \begin{cases} 1 & \text{if the state peak } x_t \in u_i \to y_j \\ 0 & \text{otherwise} \end{cases} \qquad t=1,\ldots,n$$

Definition 7-3-2 (TAR-85a)

It is said that vector Z_{ij} dominates vector Z_{qr} if, when we have $z_{ij}^t = 1$, then $z_{qr}^t = 1$ except for at least one t for which we have $z_{ij}^t = 1$ and $z_{qr}^t = 0$.

Definition 7-3-3 (TAR-85a)

It is said that the state peak $x_t \in u_i \to y_j$ is separated if and only if:

$$z_{ij}^t \text{ .AND. } z_{qr}^t = \emptyset \qquad \forall\, q_r \neq ij \text{ such that } k_{qr} \in K_1$$

With these definitions, we are able to formulate the three following rules:

Rule 1

If vector Z_{ij} (corresponding to k_{ij}) dominates all vectors Z_{qr} $\forall$ qr such that $k_{qr} \in K_1$, then the feedback loop k_{ij} is sufficient to avoid type i structural modes.

In this case, we have $z^t_{ij} = 1 \; t=1,\dots,n$, i.e. all state peaks belong to $u_i \rightarrow y_j$ and therefore k_{ij} may influence any system mode.

Rule (2)

The feedback loop k_{ij} may be eliminated from a sufficient system K_1 if the associated vector Z_{ij} is dominated by one (or more) vector(s) $Z_{qr} \; \forall \; qr$ such that $k_{qr} \in K_1$. The resulting set is a sufficient set.

It is clear that if Z_{qr} dominates Z_{ij}, then the state peaks belonging to path $u_i \rightarrow y_j$ also belong to path and element k_{ij} may then be eliminated in the presence of k_{qr}.

Rule (3)

The feedback loop $k_{ij} \in K_1$ is a necessary loop in k_1 if path $u_i \rightarrow y_j$ contains a separated state peak.

According to definition (7-3-3), a separated state peak belongs only to one input-output path, i.e. $u_i \rightarrow y_j$, and therefore k_{ij} is the only element that may form with $u_i \rightarrow y_j$ a highly connected component containing this state.

With these rules we propose (see TAR-85a), the following steps /138 for reducing the sufficient set of feedback loops K_1:

1 - Using the graph associated with the system or with the accessibility matrix:

1-1 Determine sufficient set K_1,

2 - Perform $K^* = \{ \emptyset \}$

3 - Let $K_2 \subseteq K_1$ be the subset corresponding to all separated state peaks, i.e.:

$K_2 = \{ k_{ij}$ / ij so that $k_{ij} \in K_1$ & $u_i \to y_j$ contain at least one separated state peak

If K_2 is an empty set go to 4, otherwise do the following:

3-1 Do $K^{*} = K^{*} + K_2$

3-2 If $z^t_{ij} = 1 \;\forall\; k_{ij} \in K^*$ & for $t=1,\ldots,n$ go to 7 otherwise continue

3-3 Perform $z^t_{qr} = z^t_{ij}$. OR . z^t_{qr} $t=1,\ldots,n$ for an $\neq$ qr such that $k_{ij} \in K_2$ & $k_{qr} \in K_1$

3-4 $K_1 = K_1 - K_2$

3-5 $\ell = 1$

3-6 $K_2 = \emptyset$

4 - Let $K_3 \subset K_1$ be the set of ℓ elements such that their associated vectors dominate as a set all other vectors corresponding to $k_{ij} \in K_1$. If K_3 is not empty, do $K^* = K^* + K_3$ (the existence of several sets $K_3 \to$ solution multiplicity) and go to 7. If K_3 is empty, continue.

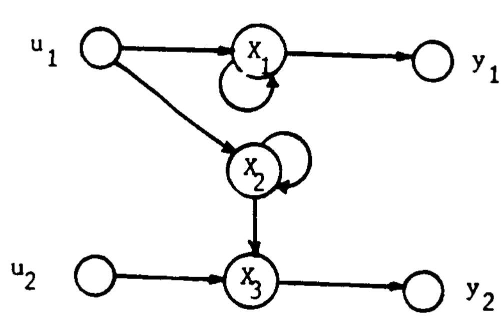
5 - Let $K_4 \subset K_1$ be the set of k_{ij} whose associated vectors are simultaneously dominated by ℓ vectors; If K_4 is not empty perform $K_1 = K_1 - K_4$ and continue.

6 - Perform $\ell = \ell + 1$ and go to 3.

7 - End: K* is a reduced set of sufficient feedback loops used for avoiding type i structural fixed modes.

Example 7-3-2

Let us consider the following example given by its associated graph:



Set K_1 is:

The corresponding vectors are:

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ z_{11} = & [1 \quad 0 \quad 0] \\ z_{12} = & [0 \quad 1 \quad 1] \\ z_{22} = & [0 \quad 0 \quad 1] \end{array}$$

States x_i and x_2 are separated (disjoined), therefore $K^* = \{k_{11}, k_{12}\}$ is a minimum solution, because z_{11} and z_{22} cover the states.

Note that the procedure does not give one solution, any more than any other minimum solution in K_1. However if step 5 is not taken into consideration (dominance elimination), we obtain all minimum solutions in set K_1.

In the above example, if we return to theorem (5-4-1), set $\{k_{21},; k_{12}\} \notin K_1$ also assures the absence of type i structural fixed modes. In effect, owing to the restriction of the set of solutions (solution $\in K_1$), the solution provided by the procedure is in general not optimum. An optimum solution contains max (I_{NS}, O_{NS}) feedback loops, where I_{NS} is the necessary and sufficient number (minimum number) of inputs for attaining all states, and O_{NS} is the necessary and sufficient (minimum number) of outputs for any state to reach an output. If the minimum number of input-output paths for covering all states is q, then the procedure provides an optimum solution if $q = \max(I_{NS}, O_{NS})$, or if these paths are totally disjoined.

The procedure is easily adaptable to the frequency range, particularly for systems with simple modes. If we let G(s) be the transfer matrix of the system and s_i $i=1,\ldots,n$ $s_i \neq s_j$ its modes, then set K_1 is given by: $K_1 = \{k_{ij}/ij$ such that $g_{ij} \neq = 0$ and the associated vectors are such that:

$$z^t_{ji} = \begin{cases} 1 & \text{if mode } s_t \text{ is a } g_{ji}(s) \text{ pole} \\ 0 & \text{otherwise} \end{cases}$$

We then continue for the formulation in the state space.

VII.3.3.b. Minimum Cost Structure

For practical applications, it is important to determine the minimum cost structures avoiding fixed modes, because structures with a large number of elements are significant only if the costs associated with the feedback loops are equal. However since costs are generally not equal (but are proportional to the distance between the control stations, for example), the determination of a minimum cost structure is of prime importance. In this section we shall formulate the problem as the well-known problem in graph /140
theory of optimizing integers (see VII.2.3b).

Given the graphic characterization of section V.4, we introduce the notion of "arc set" (TAR-85-c) associated with a state:

Definition 7-3-4 (TAR-85c)

The arc set associated with the state peak x_t is defined by $K_{x_t} = \{k_{ij}$ such that x_t belongs to a highly connected component formed by the arc (y_j, u_i) and the input-output path $u_i \to y_j\}$

The arc sets K_{x_i} may be easily determined based on accessibility matrix R. With definition (7-3-4) and theorem (5-4-1), the following corollary is direct:

Corollary 7-3-1 (TAR-85c)

If we consider a feedback matrix K, the following condition implies that the system does not have type i structural fixed modes:

$$\text{Card}\,(K^* \cap K_{x_i}) \geq 1 \qquad i=1,\ldots,n$$

where $K^* = \{k_{ij}$ such that $k_{ii} \neq$ in $K\}$

Given this corollary the problem is formulated as follows:

Find K^* such that:

$$\mathrm{Card}\,(K^* \cap K_{x_i}) \geqslant 1 \qquad i=1,\ldots,n$$

We recognize problem (7-2-1) in this problem and the solution is therefore that of the optimization problem of section VII.2.3b.

Example 7-3-3

Given example 7-3-2. The arc sets are:

$$K_{x_1} = \{k_{11}\}$$
$$K_{x_2} = \{k_{12}\}$$
$$K_{x_3} = \{k_{12}, k_{22}\}$$

Note that $K_{x_2} \subset K_{x_3}$ and therefore k_{x_3} may be eliminated; we obtain:

$$K_{x_1} = \{k_{11}\} \text{ and } K_{x_2} = \{k_{12}\}$$

The optimization problem in this case is trivial, and the solution is:

$$K^* = \{k_{11}, k_{12}\}$$

Note that the solution provided here is also sub-optimal, as in the preceding section. In the case of the example, it is optimal if $r_{11} < r_{21}$. If the costs associated with the feedback loops are equal, then the procedure provides a structure with a minimum number of elements which has the same degree of optimality as the solution provided in the preceding paragraph.

VII.3.4 Sezer's Procedures /141

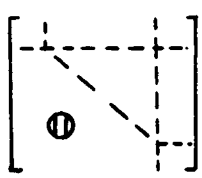
Sezer's procedure (SEZ-83) is based on Reinschke's procedure (REI-81), which enables a centralized control structure to be determined, and guarantees structural controllability and observability (see section IV.22) of a controllable and observable system (C,A,B) of order n, presented by its associated graph. He proposes the following theorem:

Theorem 7-3-1 (REI-81)

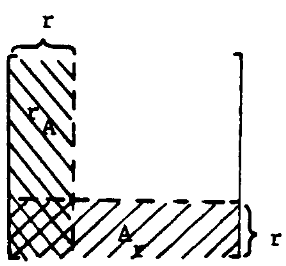
System (C,A,B) is structurally controllable and structurally observable if and only if:

1) The system is totally connectable to inputs and outputs.
2) gr (A,B) = n
3) gr $\begin{bmatrix} A \\ C \end{bmatrix}$ = n

To study the connectivity property of the system, let us assume that by permuting the lines and columns of A, it can be put in a higher block-triangular form where the diagonal blocks are irreducible matrices (the states associated with a block forming a highly connected component in the graph). To accomplish these permutations, several algorithms are available, see for example (HAR-65, KAU-68, KEV-75). Given A in the form:



Let ^rA be the submatrix containing r first left columns of A, and A_r be the submatrix containing r lower lines of A, i.e.:



The structural rank deficit of ^rA and A_r, notated rd and d_r respectively is given by:

$$\left.\begin{aligned} {}^rd &= r - gr({}^rA) \\ d_r &= r - gr(A_r) \end{aligned}\right\} \quad r=1,\dots,n$$

It is clear that ${}^{n}d = d_n = n - gr(A) = d.$ /142

Let us define the set of integers:

$$
{}^{1}r,\ {}^{2}r, \ldots,\ {}^{d}r \ \text{ such that } {}^{j}r_d = {}^{(j-1)}r_{d+1}
$$

$$
r_1,\ r_2, \ldots,\ r_d \ \text{ such that } d_{r_j} = d \quad \text{with } \overset{o}{d} = d_o = 0
$$

Example 7-3-4

$$
\begin{array}{c|c}
\begin{bmatrix} 0 & 1 & 1 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 1 \\ & & & 0 \end{bmatrix} & \begin{array}{l} {}^{d}r \\ 2 = d_4 = d \\ 1 \\ 1 \\ 1 = d_1 \end{array} \\
r_d \quad \begin{array}{cccc} 0 & 1 & 1 & 2 \\ \shortparallel & & & \shortparallel \\ {}^{1}d & & & {}^{4}d \end{array} &
\end{array}
$$

we have d = 2 and therefore ${}^{1}r = 1, \quad {}^{2}r = 4$ and

$r_1 = 1, \quad r_2 = 4$

With these definitions Reinschke (REI-81) (REI-83) gave the following theorem:

Theorem 7-3-2 (REI-81 and 83)

Let A be a higher block-triangular matrix where the diagonal blocks are irreducible, with a structural rank $gr(A) = n - d < n$. Therefore in order for the pair (A,B) to be structurally controllable, matrix B must be of dimension n x d, and:

1) elements $b_{r_i,i}$ $(i=1,\ldots,d)$ must not be not zero.

2) if the nondiagonal blocks having the same lines (i.e. $i \in I$) as a block diagonal are identically zero, then the B lines of indices $i \in I$ should contain at least one nonzero element.

If $gr(A) = n$, then minimal matrix B is a vector (nxl) for which the elements corresponding to the lowest A block should not all be zero.

It is easy to see that 1) implies that gr(A,B) = n and 2) implies the full connectivity of the system inputs.

Theorem 7-3-3 (REI-81 and 83)

Given matrix A of theorem (7-3-2), then in order for the pair $\binom{A}{C}$ to be structurally observable C must be of order (d x n) and:

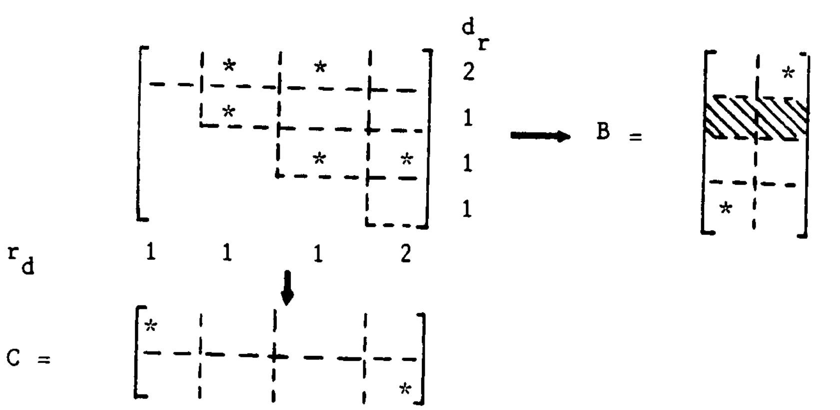
1) Elements C_{i,i_r} (i=1,...,d) should not be zero.

2) If nondiagonal blocks having the same columns, (i.e. j $\in$ J), as a diagonal block are identically zero, then the C columns of index j $\in$ J should contain at least one nonzero element.

If $gr\binom{A}{C}$ = n, then the minimal matrix C is a line /143
vector for which the elements corresponding to the first A block should not all be zero.

Example 7-3-5

For example (7-3-4), we obtain:



The shaded area should have at least one nonzero element, and * represents a nonzero element.

Remark: theorems (7-3-2) and (7-3-3) do not provide a single structure.

VII.3.4b. Sezer's Procedure: Structure With Minimal Number of Elements

Before we are able to present Sezer's procedure, a few definitions are necessary:

Definitions 7-3-5 (SER-83)

Let F be a binary matrix such that $f_{ij} = 1$ if and only if $k_{ij} \neq 0$ (F represents the structure of the control matrix under structural stress).

2 - Let $F_1 = \|f^1_{ij}\|$ & $F_2 = \|f^2_{ij}\|$ be two control structures. F_1 is said to imply F_2 if $f^1_{ij} = 1$ implies $f^2_{ij} = 1$

3 - A feasible structure is termed essential if it is not implicated by any other feasible structure.

4 - Among all feasible and essential structures, a structure is said to be minimal if it contains the smallest number of nonzero elements.

Let us recall that a system (C,A,B) has no structural fixed modes relative to F if and only if:

$$gr(M^F) = gr \begin{bmatrix} A & B & 0 \\ 0 & I_m & F \\ C & 0 & I_p \end{bmatrix} = n + m = p$$

and each state peak in the graph associated with the system belongs /144
a highly connected component with at least one arc corresponding to a feedback loop.

Below, we assume that the system is structurally controllable and observable, which implies that if gr(A) = n - d, then d ≤ min(m,p).

Let us define matrix B (C) as the matrix with the index columns (lines) $i \in I$ $(j \in J)$ of B.

Theorem 7-3-4 (SEZ-83)

Given the integer sets:

$$I = \{i_1, \ldots, i_\ell\} \qquad d \leqslant \ell \leqslant m$$
$$J = \{j_1, \ldots, j_q\} \qquad d \leqslant q \leqslant p$$
$$I' = \{i'_1, \ldots, i'_d\} \subset I$$
$$J' = \{j'_1, \ldots, j'_d\} \subset J$$

such that the system $S_{IJ} = (B_I, A, C_J)$ is structurally controllable and observable, and such that: $gr(A, B_I) = n$ and $\binom{A}{C_{J'}}$ are verified.

If F is a control structure such that gr $(F_{I,J}) = d$ and that $F_{I-J',J-J'}$ contain at least one control structure such that $F_{I-I',J-J'}$ contains at least one nonzero element in each line and column, then F is a feasible structure.

Considering theorem (7-3-4) and Reinschke's procedure, Sezer (SEZ-83) gives the following procedure to determine a feasible essential and minimal structure:

Procedure 7-3-1 (SEZ-83)

To determine a feasible essential and minimal control structure:

1 - Let I and J two index sets of an essential and minimal structure determined using Reinshcke's procedure (see VII.3.4.a)

2 - Select $I' \subset I$ and $J' \subset J$ such that: $gr(A, B_{I'}) = n$ and $gr \binom{A}{C_{J'}} = n$

3 - Construct $F_{I-I',J-J'}$ such that it contains exactly nonzero elements positioned in different lines and columns.

4 - Construct $F_{I-I',J-J'}$, such that it contains exactly max (,q)-d nonzero elements positioned so as to not leave a line or column zero.

5 - Position all other F elements at zero.

With theorem (6-3-4), F is feasible and essential, as for certain $f_{ij} = 1$, if $i \in I-I'$ and $j \in J-J'$, then the link loss (y_i, u_i) leaves a highly connected component, in the graph associated with the system, not containing a feedback matrix element. Conversely, if $i \in I$ and $j \in J$ then since I and J are minimal, $(A, B_{I-\{i\}})$ and $(a, C_{J-\{j\}})$ are structurally uncontrollable and unobservable. In both cases and without f_{ij}, F is not feasible. Finally, F is an essential and minimal structure just as I and J are minimal. /145

A direct consequence of this procedure is that the number of minimum elements of an essential and minimal structure is max (,q) where and q are the minimal input and output numbers which guarantee the controllability and observability of the system.

Example 7-3-6 (SEZ-83)

Given the following system:

$$\overset{\circ}{X} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} X + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reinschke's procedure gives: $I = J = \{1,2\}$ and $I' = J' = \{1\}$ from which we obtain, using Sezer's procedure, the single solution $F = \text{diag}(1,1)$.

Note that for this example, the structure:

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

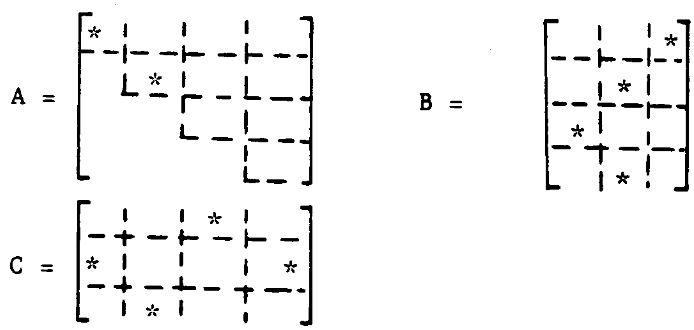
is also feasible and minimal (it may be verified by plotting the graph associated with the system). Consequently, the procedure generally does not provide all feasible solutions. The above example shows that a feasible essential and minimal solution does not necessarily contain the feedback links d in Y_j $j \in J$ and U_i $i \in I$. This situation arises when we have:

$$gr \begin{bmatrix} A & B_I \\ C_J & 0 \end{bmatrix} = n + \min(\ell + q) \qquad (7\text{-}3\text{-}3)$$

in this case we have: gr $(M^F) = n + m + p$ for any F with $gr(M^F) = n+m+p$ for any F with $gr(F_{IJ}) = \min(f\ ,q)$

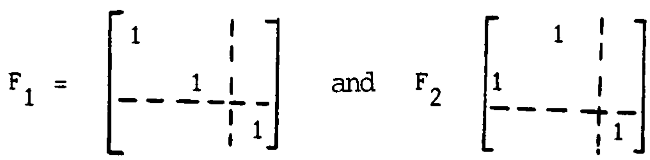
Example 7-3-7 (SEZ-83)

Given system (C,A,B) with:



For which we have a: I = J {1,2,3} and I' = J' = {1,2}. Sezer's procedure gives the two following structures:

/142



Note, in contrast to the preceding example, that (7-3-3) is not verified in this case, because there is no F that verifies $gr(M^F) = n+mF+p$, unless $gr(P_{I,J}) = d = 2$, even if $gr(F) = \min(\ell,q) = 2$.

Therefore, the solutions provided by the procedure are the only feasible, essential and minimal ones.

Sezer's procedure itself does not require calculation, but it is based on Reinschke's procedure which needs to "block-triangularize" the system and calculate the structural rank of 2N submatrices to determine the subsets I:, J, I' and J' required for Sezer's procedure. In short, despite this calculation and although the procedure generally does not provide all feasible solutions, Sezer's method is still of interest because it still proposes at least one structure with a minimum number of elements. Finally, note that this procedure does not give any indication of link costs and from this standpoint the solution set forth generally remains suboptimal.

VII.3.5. Comments

The algorithms presented in this section (with the exception of Senning's) do not provide a control structure for which the system does not have fixed modes and the gain determination should be done in later (see ch. VIII).

Locatelli et al's approach (LOC-77) is valid for systems with simple modes and its solution is generally suboptimal. The approaches of section VII.3.4. are only valid for avoiding structural fixed modes. The advantage of the approach of section VII.3.4a (which we propose) is its simple formulation, but it only makes it possible to avoid type i structural fixed modes and the solution is generally suboptimal. Sezer's approach requires many calculations, but it provides a structure with a minimum number of elements avoiding structural fixed modes. Senning's procedure seems to be the most interesting of these approaches, because it generally includes the gain structure and value without having to first test for the existence of fixed modes.

Arbitrary pole placement of a system is mathematically equivalent to an arbitrary placement of coefficients $a_1, \ldots, a_n$ of its characteristic polynomial:

$$\det(sI-A-BKC) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n.$$

These coefficients depend on K elements, which are considered here to be the degree of freedom of the problem. Theory (5-5-1) determines these coefficients from the graph associated with the system using the "circuit family" notion (see def. 5-5-1). It is evident that an arbitrary placement of these coefficients requires n feedback loops.

Based on the theory of real functions with several variables Reinschke (REI-83 and 84b) concludes that an arbitrary pole placement necessitates that the matrix of dimension n x (mxp):

$$\begin{bmatrix} \frac{\partial a_1}{\partial k_1} & \frac{\partial a_1}{\partial k_2} & \cdots\cdots \\ & & \\ \frac{\partial a_n}{\partial k_1} & \frac{\partial a_n}{\partial k_2} & \cdots\cdots \end{bmatrix}$$

has a rank equal to n, where k_1, k_2,... are K elements.
One immediate consequence is that if n > mxp then an arbitrary pole placement is impossible.

Theorem 7-4-1 (REI-83 and 84b)

One necessary and structurally sufficient condition for an arbiltrary pole placement of the system is the existence of n circuit families of width 1,2,...,n in the graph associated with the system (see V.5), where each family contains an arc corresponding to a different feedback loop.

Example 7-4-1

Let us consider the system of example 7-3-2, and a full feedback matrix. Fig. (7.1) gives the related graph. One possibity for satisfying theorem (7-4-1) is to put $k_{21} = 0$, fig. (7.2) gives the circuit families for this choice: /148

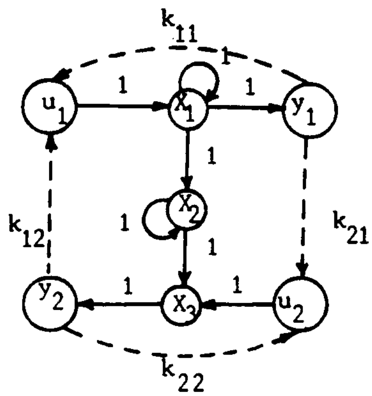


Fig. 7.1

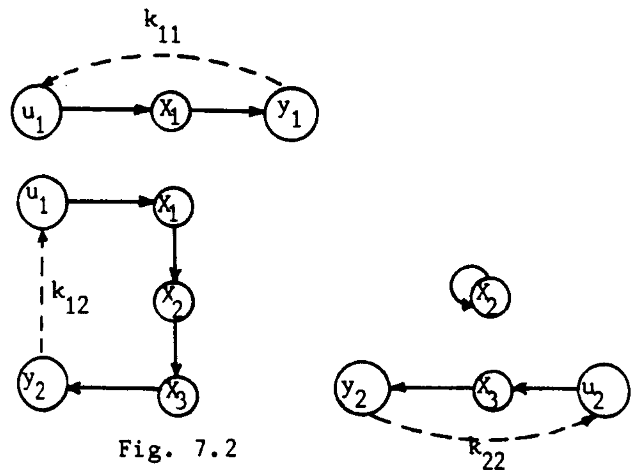


Fig. 7.2

Recently Evans and Kruser (EVA-84) provided this result in terms of a rank test of a binary matrix: a necessary and sufficient condition for these poles of a system to be structurally assignable est that: $\mathrm{TR}\ (M_{KL}\ \mathrm{V.\Lambda}\ \ M_{La}) = n$, $\mathrm{V.\Lambda}$ is the Boolean product and: M_{KL} and M_{La} are to binary matrices given by:

$$\begin{array}{c} \\ L_1 \\ \vdots \\ \vdots \\ L_q \end{array} \begin{array}{c} a_1 \ \cdots\cdots \ a_n \\ \left[\begin{array}{ccc} 1 & \cdots\cdots & \\ 1 & & \\ \vdots & & \\ \vdots & & \end{array}\right] \end{array} = M_{La} \quad \& \quad \begin{array}{c} \\ k_1 \\ k_2 \\ \vdots \\ \vdots \end{array} \begin{array}{c} L_1 \ \cdots\cdots \ L_q \\ \left[\begin{array}{ccc} 1 & \cdots\cdots & \\ 1 & & \\ \vdots & & \\ \vdots & & \end{array}\right] \end{array} = M_{KL}$$

where $a_1, \ldots, a_a$ are coefficients of the characteristic polynomial, k_1, $k_2, \ldots$ are the nonzero elements of K and $L_1, \ldots,$ L_q the circuits in the graph associated with the system containing arcs corresponding to feedback loops. $(m_{KL})_{ij} = 1$ if and only if the k_i element belongs to circuit L_j and $(m_{La})_{ij} = 1$ if and only if the L_i circuit belongs to the a_j coefficient (see theorem 5-5-1).

VII.5 CONCLUSION

In this chapter we have shown the different methods of determining a control structure that avoids fixed modes. We think that the approach of relieving structural stresses is the most natural one for eliminating fixed modes. Most of the corresponding methods only determine the structure, with the exception of that of Senning who completely determines the control (structure and gains) /149

and this is why it is of interest. The feedback gains should be determined in a later stage. This is the subject of the next chapter.

Despite the few studies available for decomposing the system into subsystems avoiding fixed modes, it seems to us that the problem of searching for control structures (dynamic or static) at minimum cost should be investigated more thoroughly and therefore remains an important research area.

CHAPTER VIII - SUMMARY OF DECENTRALIZED CONTROLS IN THE ABSENCE OF FIXED MODES

VII.1 INTRODUCTION

We have seen in chapter II that the fundamental result of a decentralized fixed mode is that the absence of unstable fixed modes is a necessary and sufficient condition for the system to stabilizable in a decentralized manner. Therefore if the system does not have fixed modes or if the fixed modes are sufficiently stable, the problem consists of determining the control gains to assure good system performance. This chapter focuses on the different methods of determining these gains, while assuring a certain optimality in terms of the traditional quadratic criterion. We focus more particularly on the parametric optimization approach, /152
as we propose an algorithm for calculating a robust control based on this technique. In effect, the algorithm proposed uses the

gradient method projected and accordingly it may considered as a modification of Germomel's and Bernussou's algorithm (GER-79a).

VII.2 PARAMETRIC OPTIMIZATION SYNTHESIS

VIII.2.1. Optimization Problem

Let us consider an invariant system, composed of N subsystems and represented by the following global state equation:

$$\overset{\circ}{X}(t) = A\, X(t) + B\, U(t)$$
$$Y(t) = C\, X(t)$$

where $x \in R^n$, $U \in R^m$ & $Y \in R^p$ are the state, the controls and the outputs of the system, respectively. Let us assume that we only have measurements on part of the state, i.e. that it is necessary for a control to be created by a static output feedback(*) with time-invariant gains:

$$FU = -KY = -KCX$$

The problem is therefore formulated as follows:

$$\min_K J(K) = \int_0^\infty (X^T Q X + U^T R U)\, dt$$

$$\text{under} \quad \overset{\circ}{X} = AX + BU$$
$$Y = CX \qquad (8\text{-}2\text{-}1)$$
$$U = -KY$$

with Q, R symmetrical matrices defined as semi-positive and positive respectively, i.e. $Q \geq 0$ and $R > 0$.

The closed loop system is given by:

$$\overset{\circ}{X}(t) = (A - BKC)\, X(t) = D\, X(t)$$

the solution to this equation is:

$$X(t) = e^{(A-KBC)t}\, X(o) = T(t,o)\, X(o) \qquad (8\text{-}2\text{-}2)$$

where T(t,o) is the transition matrix of the closed loop system. Let us replace (8-2-2) in the expression of the criterion; we obtain:

(*) The synthesis of dynamic controls is reduced to the synthesis of static controls (see appendix 5).

$$J(K) = \int_o^{\infty} X^T(o)\ T^T(t,o)\ Q_1(K)\ T(t,o)\ X(o)\ dt$$

with $Q_1(K) = Q + C^T K^T R K C$

Using a property of the plotting of a matrix $tr(b^T a) = tr(ab^T)$ leads us to: /153

$$J(K) = Tr\ (P\ X_o)$$

with

$$X_o = X(o)\ X(o)^T$$

$$P = \int_o^{\infty} T^T(t,o)\ Q_1(K)\ T(t,o)\ dt$$

where P is the solution to Lyapunov's equation:

$$D^T P + PD + Q_1(K) = 0$$

with D the dynamic closed loop matrix D = A-BK. The problem then takes on the following form:

$$\min_K J(K) = Tr\ (P\ X_o)$$

$$\text{under } f(K) = D^T P + PD + Q_1(K) = 0 \qquad (8\text{-}2\text{-}3)$$

Let us define the set of feasible solutions by:

$$K_c = \{ K\ /\ K \in R^{mxp} \quad \text{and such that } F(K) = 0\} \qquad (8\text{-}2\text{-}4)$$

where F(K) represents the decentralization stress applied, given by:

$$F(K) = K - \text{block.diag}(K_1, \ldots, K_N) \qquad (8\text{-}2\text{-}5)$$

The problem becomes:

$$\min_K J(K) = Tr\ (P\ X_o)$$

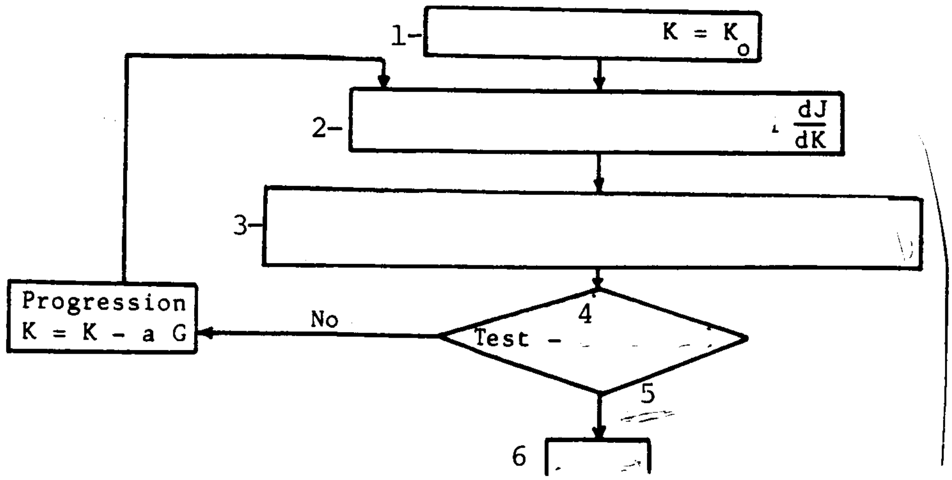
$$\text{under } f(K) = D^T P + PD + Q_1(K) = 0 \qquad (8\text{-}2\text{-}6)$$

$$F(K) = 0$$

with D, $Q_1(K)$, X_o & $F(K)$ defined above.

VIII.2.2. Projected Gradient Method - Geromel's and Bernussou's Method

To solve problem (8-2-6) and obtain a solution satisfying the structural stresses applied, Geromel and Bernussou (GER-79a, 79c, 82) propose the use of a "primal" "projected gradient" method which is summarized by the flow chart of figure (8.1).



Key: 1-Initialization; 2-Matricial gradient calculation; 3-Projections over set defined by structural stresses: G direction; 4-Test - Stability; 5-Yes; 6-Stop.

VII.2.2a. Calculation of the Matricial Gradient /154

The matricial gradient of J(K) is easy to calculate using the method given in (BER-81) for calculating the matricial derivative of a composed function and which is summarized by the following theorem for a static case (for a dynamic system see (GER-79c)):

Theorem 8-2-1 (BER-81)

Given the composed scalar function (f(X,G(X)); if f is derivable with respect to all arguments (X and G(X) being matrices) then:

$$\frac{df(X,G(X))}{dX} = \frac{\partial L(X,Y^*,Z^*)}{\partial X}$$

where L(X,Y,Z) is the Lagrangian given by:

$$L(X,Y,Z) = f(X,Y) + \mathrm{Tr}\left[Z^T(G(X) - Y)\right]$$

Y* and Z* are solutions of the steady conditions:

$$\frac{\partial L(X,Y,Z)}{\partial Y} = \frac{\partial L(X,Y,Z)}{\partial Z} = 0$$

which gives: $L = Tr\,(P\,X_o) + Tr\,[S^T f(K)]$

$$\frac{dJ}{dK} = 2\,(RKC - B^TP)\,SC^T \tag{8-2-7a}$$

with

$$f(K) = D^TP + PD + Q_1(K) = 0 \tag{8-2-7b}$$

$$g(K,X_o) = DS + S\,D^T + X_o = 0 \tag{8-2-7c}$$

$$D = A - BKC \tag{8-2-7d}$$

$$Q_1(K) = Q + C^T K^T R\,KC \tag{8-2-7e}$$

To calculate gradient (8-2-7a) it is necessary to solve Lyapunov's two equations. This is a reasonable task if the order of the system is not very high. Note that the two equations (8-2-7b) and (8-2-7c) are decoupled for a known D and and they may therefore be solved concurrently.

If we consider that all states of the system are accessible for measurements (state feedback control) then: $u = -KX$ and the (8-2-7c) equations are expressed in this case:

$$\frac{dJ}{dK} = 2\,(RK - B^TP)\,S \tag{8-2-8a}$$

with /155

$$f(K) = D^TP + PD + Q_1(K) = 0 \tag{8-2-8b}$$

$$g(K,X_o) = DS + S\,D^T + X_o = 0 \tag{8-2-8c}$$

$$D = A - BK \tag{8-2-8d}$$

$$Q_1(K) = Q + K^T R\,K \tag{8-2-8e}$$

The optimal control is given by:

$$K^* = R^{-1}\,B^T\,P \tag{8-2-9}$$

where P is the solution to Ricatti's equation:

$$A^TP + P\,A - P\,B\,R^{-1}\,B^T\,P + Q = 0 \tag{8-2-10}$$

Here equation (8-2-8c) has become useless and the control no longer depends on the initial conditions.

VIII.2.2b. Projection of the Gradient - Feasible Direction

Let us consider the feasible gains K_c defined in (8-2-4), where F(K) is a linear function of K and F(o) = 0. Let us assume that $K \in D_c$ is an initial solution such that $\frac{dJ}{dK} \neq 0$, then the feasible direction G, for which a progression set exists such that J(K-aG) J(K) for any $0 < a \leqslant \bar{a}$ & $\bar{a} > 0$, through orthogonal projection of the matricial gradient $\frac{dJ}{dK}$ on the hypersurface representing the stresses. Such a projection may be realized by solving the following optimization problem (GER-79c):

$$\min_{G} \tfrac{1}{2} \operatorname{Tr} \{(\frac{dJ}{dK} - G)^T (\frac{dJ}{dK} - G)\} \qquad (8\text{-}2\text{-}11)$$

Under $F(G) = 0$

The Lagrangian of this problem is expressed:

$$L(G,E) = \frac{1}{2} \operatorname{Tr} \{(\frac{dJ}{dK} - G)^T (\frac{dJ}{dK} - G)\} + \operatorname{Tr} \{E^T F(G)\}$$

The stationarity conditions give:

$$\frac{\partial L}{\partial G} = 0 \longrightarrow G - \frac{dJ}{dK} + \frac{\partial L}{\partial G} \operatorname{Tr} \{ E^T F(G)\} = 0$$

$$\frac{\partial L}{\partial E} = 0 \longrightarrow F(G) = 0$$

To verify the decentralization stress G should have a diagonal structure G_d and the stationarity conditions are expressed:

$$G - \frac{dJ}{dK} + E = 0 \qquad G \in G_d$$
$$\qquad\qquad (8\text{-}2\text{-}12)$$
$$\operatorname{Tr} (E^T G) = 0 \qquad G \in \bar{G}_d$$

where $\bar{G}_d$ is the complementary set of G_d ($G_d \cup \bar{G}_d = R^{mxp}$ & $G_d \cap \bar{G}_d = \emptyset$).

The solution to (8-2-12) gives the expression of the projection /156
operator which, in this case, is expressed by:

$$\Pi_d(\cdot) = \text{block.diag}\{(\cdot)_1,\dots,(\cdot)_N\}\ ; \quad (\cdot) \in R^{m_i,p_i} \qquad (8\text{-}2\text{-}13)$$

Therefore, the projection over the decentralization set simply consists of eliminating the non block-diagonal terms from the expression of the matricial gradient.

If the structural stress applied were more general than the full decentralization, i.e. K_F, the stationarity conditions (8-2-6) would remain the same provided that G_d is replaced by G_F and $\bar{G}_d$ by $\bar{G}_F$. Consequently, the projection of the matricial gradient over the set of matrices under any structural stress simply consists of removing the terms from the gradient expression corresponding to the zero terms of the stress.

The existence of a feasible direction matrix is guaranteed by the following lemma:

Lemma 8-2-1 (GER-82)

The optimal solution to the projection problem (8-2-11), with K_c given by (8-2-4), is such that:

a) if $\|G\| \neq 0$ then $\bar{a} > 0$ exists such that $J(K-aG) < J(K)$

$$\forall \quad 0 < a \leqslant \bar{a}$$

b) if $\|G\| = 0$ matrix K satisfies Kuhn-Tucker's conditions of the problem

$$\min_{K \in K_c} J(K)$$

Note that this lemma only shows the existence of a value a such that K = aG is stabilizing. To speed up the convergence of the algorithm, it is desirable that a is as high as possible. It may therefore be obtained by solving the problem:

$$\min_{a \geqslant 0} J(K-aG)$$

However this problem has no analytical solution, and it is better to adopt an heuristic method of determining a for each iteration.

VIII.2.2c. Stability of the Algorithm

For each iteration, the algorithm provides a K matrix which plays the role of the initial matrix for the following iteration. Accordingly, and to guarantee the stability and convergence of the algorithm, matrix K should be such that the closed loop system is asymptotically stable, therefore $K \in K_s$; K_s defined by:

$$K_s = \{ K / K \in K_c \text{ and that the closed loop system is asymptotically stable} \} \quad (8\text{-}2\text{-}14)$$

The following lemma gives the sufficient conditions in order /157
for the algorithm to create a stabilizing control for each iteration.

Lemma 8-2-1 (GER-79c)

If $J(K)$ is a differentiable function relative to K, then $\forall K \in K_s$:

a) For each iteration there is a step $a > 0$ such that:

$$J(K-aG) < J(K) \text{ or } G \;\; \Pi_d \left(\frac{dJ}{dK}\right)$$

when $G \neq 0$

b) If the pair $(A, Q^{\frac{1}{2}})$ is completely observable and $X_o > 0$ therefore $\forall a$ such that

$$J(K-aG) < J(K) < \infty \rightarrow (K-aG) \in K_s.$$

This lemma gives a sufficient condition for the existence of a such that K - aG is stabilizing. For each iteration, it should therefore be ascertained whether the gain obtained is stabilizing, therefore a should be such that:

i) $(K - aG) \in K_s$

ii) $Tr \{P(K-aG) \, X_o \} < Tr \{P(K) \, X_o \}$

When the algorithm requires the calculation of matrix P, then condition i) may be verified by testing the definition of P.

Actually Geromel and Bernussou (GER-79a) use the the following conventional adaptive law for determining a:

C-3

$a^{i+1} = \pi a^i$ if $J(K^i - a^i G^i) < J(K^i)$ and $P(K^i - a^i G^i) > 0$

$a^{i+1} = \nu a^i$ otherwise

VIII.2.2d. Degree of Suboptimality

The problem of nonlinear optimization considered here is generally nonconvex with respect to K. Therefore Geromel's and Bernussou's algorithm, which satisfies the necessary optimality conditions, tends toward a local minimum or an inflection point in space of parameters k_{ij}. Of course the convergence point depends on the initial solution K_0 (step 1 of the algorithm). The degree of suboptimality of the solution is defined by where J* corresponds to the optimal solution without structural stresses as given by (8-2-9) and J is the optimal solution obtained by the algorithm presented.

VIII.2.2e. Initialization of the Algorithm - Armentano's and Singh's Algorithm

Like all other "primal" methods (projected gradient), Geromel's and Bernussou's algorithm (GER-79a and c) requires an initial stabilizing gain matrix satisfying the stresses applied. The existence of this gain is related to the existence of fixed modes /158
relative to the stresses applied (see ch. II). In literature, several studies are found, based on conjectures, for determining this initial gain (AOK-73), WAN-78b, FES-79, IEK-79, SEZ-81c) which are discarded one after the other by counter examples. Among those which give procedures for determining this gain, let us mention (DAV-76Aa, ARM-81). We will present Armentano's and Singh's algorithm (ARM-81) because, by this algorithm, a solution can always be obtained as long as there are no unstable fixed modes. Further, the other algorithms additionally require local controllability i.e. the controllable pair (A_{ii}, B_i).

Armentano and Sing (ARM-81) use an approach already proposed by Mc Brinn and Roy (MCB-72) to calculate an output feedback control without structural stress. This approach is based on the calculation of the gradient of the natural dominant value of the closed loop system. We have seen in chapter III that for distinct natural values, this gradient is given by:

$$\frac{\partial s}{\partial k_{ij}} = \begin{cases} - W^T b^i c_j V & \text{if } k_{ij} \neq 0 \\ 0 & \text{if } k_{ij} = 0 \end{cases}$$

where V and W are the left and right natural vectors of the dynamic closed loop dynamic matrix. The real part of the gradient is therefore given by:

$$g_{ij} = \text{Re} \left(\frac{\partial s}{\partial k_{ij}}\right) \qquad (8\text{-}2\text{-}15)$$

Use of (8-2-15) for the dominant natural value of the closed loop system enables Armentano and Singh to propose the following procedure:

Step 1: Arbitrarily select a gain satisfying the stresses i.e.: $K \in K_d$

Step 2: Calculate the dominant natural value s_d of the dynamic closed loop matrix (A-BKC) if s_d is negative (or its real part is negative) stop. Otherwise, calculate the natural right and left vectors of (A-BKC) associated with s_d.

Step 3: Calculate the gradient matrix G defined in (8-12-15);

Step 4: Drop down on the line by more than a large slope (direction - G); do $K \leftarrow K-aG$ (a: no iteration) go to step 2.

Remarks

1 - If the dominant natural value (with a positive or zero real part) tends toward a local minimum, then the algorithm should recommence with another gain.

2 - A unidirectional search for a may be improved using a quadratic interpolation. /159

3 - If the dominant natural value becomes multiple during each iteration, disturb K slightly (WAN-73a) to make it a single value again.

VIII.2.2.2f. Flow Chart and Example

In short, Geromel's and Bernussou's algorithm may be presented by the flow chart of figure. 9.2

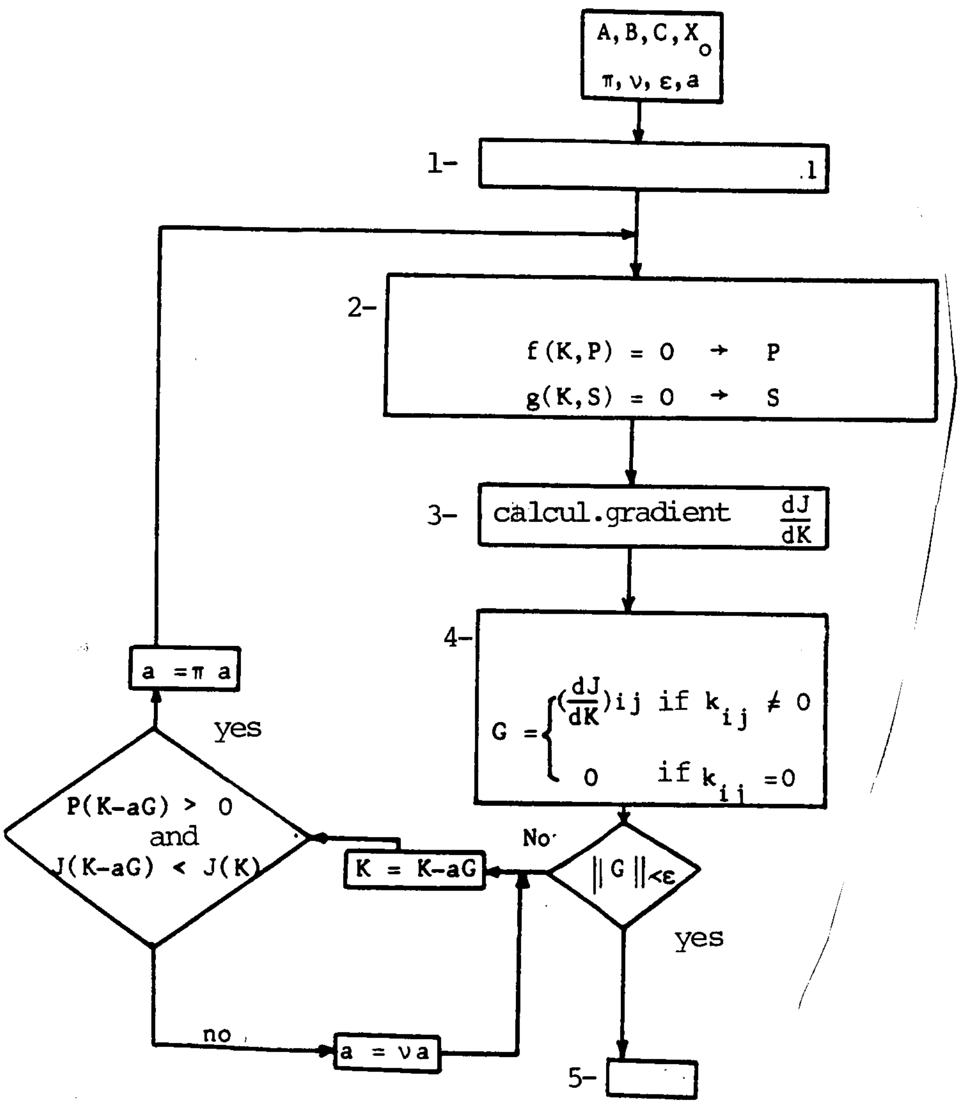


Fig. 8.2 - Flow Chart of Geromel's and Bernussou's Algorithm

Key: 1-Calculation of initial gain; 2-Solve Lyapunov's equations; 3-Calculate gradient ; 4-Projection of gradient; 5-Stop.

Example 8-2-1 (GER-79a and c)

Let us consider the following system (with three subsystems N=3): $\overset{\circ}{X} = A\,X + B\,U$

with: /160

$$A = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0,5 & 1 & 0,6 & 0 \\ -2 & -3 & 1 & 0 & 0 & 1 \\ \hline 0,5 & 1 & 0 & 2 & 1 & 0,5 \\ 0 & 0,5 & 1 & 3 & 0 & -0,5 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0,5 & 0,5 & 0 & -3 & -4 \end{array}\right] \qquad B = \left[\begin{array}{c|cc|c} 1 & & & \\ 1 & & & \\ \hline & 3 & 0 & \\ & 0 & 4 & \\ \hline & & & 2 \\ & & & 2 \end{array}\right]$$

For a state feedback control:

$$U_i = -\,K_i\,X_i \qquad i=1,2,3$$

and for matrices $Q = I_6$ and $R1 = I_4$; the initial gain retained (calculated using Geromel's and Bernussou's approach (GER-79b) based on Lyapunov's theory) is:

$$K^1 = \left[\begin{array}{cc|cc|cc} 11,51 & -\,0,55 & & & & \\ \hline & & 2,3 & 0,62 & & \\ & & 0,82 & 3,41 & & \\ \hline & & & & 4,96 & 0,06 \end{array}\right] \qquad (8\text{-}2\text{-}16)$$

with $a^1 = 0.1$, $\pi = 1.2$, $\nu = 0.5$ and $X_0 = I$, the application of Geromel's and Bernussou's algorithm, programmed with double precision on an IBM 370/165, for an end of calculation test $\varepsilon = 10^{-3}$ gives, after 59 iterations, the following optimal gain:

$$K^{59} = \left[\begin{array}{cc|cc|cc} 1,21 & 0,35 & & & & \\ \hline & & 1,43 & 0,45 & & \\ & & 0,87 & 2,46 & & \\ \hline & & & & 1,58 & 0,24 \end{array}\right]$$

the initial value of the criterion is $J(K^1) = 8.76$ and at convergence $J(K\ ) = 3$. The natural values of the closed loop system are:

$$\{-1.7262 \mp 1.2784,\ -4.2746 \mp 0.8277,\ -6.0666,\ -5.2803\}.$$

VIII.2.3. Chen's Mahmoud's and Singh's Iterative Procedure

Chen et al (CHE-84) considered the control problem by decentralized state feedback, i.e.:

$$\min_{K \in K_d} \operatorname{Tr}(P X_o)$$

$$\text{sub} \quad f(K) = D^T P + PD + Q_1(K) = 0$$

$$D = A - BK$$

$$Q_1(K) = Q + K^T R K$$

The solution to this problem without structural stress is given by:

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$$K^* = R^{-1} B^T P \qquad (8\text{-}2\text{-}9)$$

where P is the solution of Riccati's equation as follows:

$$A^T P + PA - P B R^{-1} B^T P + Q = 0 \qquad (8\text{-}2\text{-}10)$$

To obtain a suboptimal solution under structural stresses, Chen et al (CHE-84) propose a modification of Geromel's and Bernussou's algorithm as follows: Based on an initial full gain matrix (given centralized optimal solution (by 8-2-9)), the blocks which should be zero are successively cancelled and the optimal matrix is searched for during each cancellation using the gradient method. This procedure is summarized by the following steps:

Step 1: Solve Ricatti's equation (8-2-10) to calculate the initial gain matrix: $K^1 = R^{-1} B^T$ i.e.

$$K^j = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & K_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{bmatrix}, \quad j=1$$

Step 2:

a) Cancel a k^j block, i.e. K_{IN} to define:

$$K^i = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1,N-1} & 0 \\ \vdots & \ddots & & & K_{2N} \\ \vdots & & \ddots & & \vdots \\ K_{N1} & \cdots & \cdots & \cdots & K_{NN} \end{bmatrix}$$

b) Calculate the natural values of $(A-B\ K^i)$: if matrix $(A-B\ K^i)$ is stable go to step 3, otherwise consider K_i as the starting matrix of Armentano's and Singh's algorithm (see VIII.2.2) and find a matrix K^i which stabilizes matrix $(A - B\ K^i)$.

Step 3: Using matrix k^i, solve Lyapunov's two equations (using Hoskin et al's method (HOS-77) for example) (8-2-8b) (8-2-8c) to calculate the gradient (8-2-8a). Using a conjugate gradient algorithm, with a unidirectional search, using a quadratic interpolation, determine a matrix K^i which minimizes the criterion.

Step 4: Cancel another K^i block, i.e. K_{N1} we obtain K^{m+1} /162

$$K^{i+1} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1,N-1} & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ K_{N-1,1} & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & K_{NN} \end{bmatrix}$$

put i   i+1 and go to step 2.

The procedure continues until the desired block-diagonal structure is obtained.

EXAMPLE 8-2-2 (CHE-84)

Let us consider example (8-2-1); application of this procedure gives:

$$K^1 = \left[\begin{array}{cc|cc|cc} 0.9004 & 0.2487 & 0.1887 & 0.1823 & 0.3117 & 0.04 \\ \hline 0,2761 & 0,29 & 1,143 & 0,4902 & 0,4006 & 0,1202 \\ 0,4608 & 0,2684 & 0,6536 & 2,219 & 0,2878 & -0,031 \\ \hline 0,6068 & 0,1366 & 0,3873 & 0,1207 & 1,184 & 0,32 \end{array}\right]$$

After 6 iterations we obtain:

$$K^6 = \left[\begin{array}{cc|cc|cc} 1,145 & 0,3162 & & & & \\ \hline & & 0,1264 & 0,4981 & & \\ & & 0,7882 & 2,2224 & & \\ \hline & & & & 1,366 & 0,2269 \end{array}\right]$$

with $J(K^1) = 2.718$ & $J(K^6) = 3{,}007$

Remarks

1 - The number of iterations is at the most equal to $N(N-1)$, but each iteration requires a fairly large number of subiterations according the number of zero elements in the removed block.

2 - There are no restrictions in the block cancellation sequence. The procedure is applicable, for any structural stresses, without modification.

3 - The procedure may be adapted to calculate an output feedback control. In effect, if C^{-1} exists the only change is in the given initial gain value, and in this case by: $K = R^{-1} B^T P C^{-1}$ (P solution of Ricatti's equation (8-2-10) and replacing equations (8-2-8) by (8-2-7).

If $\operatorname{rank}(C) < n$, then C^{-1} does not exist and the proce- /163
dure requires the calculation of an optimal initial gain $K = R^{-1} B^T P C^T (C C^T)^{-1}$ this gain may be found by applying step 3 of the procedure.

4 - According to Chen et al (CHE-84), this procedure is more

advantageous than Geromel's and Bernussou's algorithm because it requires less computer time: for example Chen et al inserted two algorithms on PDP-10 and the CPU time required for example (8-2-1), was: (CHE-84):

Geromel's and Bernussou's Algorithm

Initial block-diagonalization	53.2 sec
Suboptimal block-diagonalization	101.26 sec

Chen et al's Procedure

Initialization (full matrix)	2.56 sec
Suboptimal block-diagonalization	35.81 sec

5 - Like Geromel's and Bernussou's algorithm, the procedure gives for all iterations a stabilizing gain and the structure of the feasible direction verifies the stress for this same iteration.

VIII.2.4. Geromel's and Peres' Iterative Procedure

Geromel and Peres (GER-84) also considered the case of a decentralized state feedback control and introduce stresses like linear stresses on the set of control matrices. The set of feasible control matrices is given by:

$$K_c = \{ K \ / \ K \in R^{mxp} \quad \text{and such that } F(K) = 0\} \qquad (8\text{-}2\text{-}4)$$

where F(K) represents the stress applied, given by:

$$F(K) = K\ H = K[I - C^T\ F(C\ C^T)^{-1}\ C] \ (8\text{-}2\text{-}17)$$

for the case of an output feedback, and by:

$$F(K) = K = \text{block diag}\ (K_1, \ldots, K_N) \qquad (8\text{-}2\text{-}18)$$

for the decentralization stress.

Let us recall that the optimal control by state feedback is given by:

$$K^* = 1R^{-1} B^T P \qquad (8\text{-}2\text{-}9)$$

where P is the solution to Riccati's equation:

$$A^T P + PA - P B R^{-1} B^T P + Q = 0 \qquad (8\text{-}2\text{-}10)$$

we may express $\pi(p) = 1Q$ (8-2-19)

To solve the problem iteratively, Geromel and Peres show the following property:

If matrix K satisfies

$$K + L = R^{-1} B^T P \qquad (8\text{-}2\text{-}20)$$

where P is a matrix defined as being positive, and a solution to Riccati's equation /164

$$\pi(p) = Q + L^T R L$$

then matrix (A - BK) is asymptotically stable for any arbitrary matrix L.

If L = 0, we recognize equation (8-2-19) and therefore the optimal decentralized control (8-2-9). However since L is completely arbitrary, then it may be selected in such a manner to have K satisfying the stress applied. In effect it is easy to see that according to (8-2-20) and (8-2-4) and for stress F(K) defined by (8-2-17) or (8-2-18), that matrix L should be:

$$L = F\,(R^{-1} B^T P)$$

Finally the degree of suboptimality of the solution is defined by:

$$d_{so} = \frac{J(P) - J(P^*)}{J(P^*)}$$

where $J(P)$ is the value of the criterion at convergence and $J(P^*)$ the value of the criterion corresponding to the centralized optimal solution ($L = 0$). In order for d_{do} to be independent of the initial conditions, it is assumed that $X(o)$ is a Gaussian random variable with a mean value of zero and whose variance is equal to one, we therefore obtain:

$$d_{so} = \frac{Tr(P) - Tr(P^*)}{Tr(P^*)}$$

With this definition and the above property, Geromel and Peres (GER-84) propose the following iterative procedure for calculating a decentralized suboptimal control:

Step 1: $i = 0, \quad L_i = 0$

Step 2: Determine P_i by solving Ricatti's equation:

$$\pi(P_i) = Q + L_i^T \quad R \quad L_i$$

if $i = 0$ calculate $J^* = Tr(P_o)$

Step 3: Calculate by (8-2-16) or (8-2-17): $L_{i+1} = F(R^{-1} B^T P_i)$

Step 4: If $\|L_{i+1} - L_i\| < \varepsilon$, where ε is a small positive real, go to step 5, otherwise, put $i \leftarrow i+1$ and return to 2.

Step 5: Calculate $J = Tr(P_i)$ and $K \in K_c$ by:

$$K = R^{-1} B^T P_i - L_{i+1}$$

The suboptimality ratio is given by: $d_{so} = \frac{J - J^*}{J^*}$

Remarks: /165

1 - It is clear that the performance of the algorithm depends on the method of solving Ricatti'ss equation in step 2. One interesting method is given in (KLE-68F).

2 - At convergence we have:

$$\pi(P_i) = Q_i = Q + L_i^T R L_i$$

since Q_i is a positive semi-defined matrix, the problem may then be seen as a quadratic linear problem for determining the state penalty matrix to have $K \in K_C$. Since $Q_i \geq Q$ the criterion will increase: this is the price to pay to satisfy the stresses on the control.

3 - the feedback matrix obtained is independent of the initial conditions such that the closed loop system is asymptotically stable.

4 - It is easy to account for a general structural stress represented by K_F (in this case we have $F(K) = K - K_F$) and even any linear stress $F(K)$.

VIII.2.5. Comments

All methods presented in this section use the global model of the system, and are based on an optimization of the parameters in space. They require the resolution of Lyapunov's or Riccati's equations of the same order as the system. They will therefore have numerical type problems for systems with very large dimensions. Additionally, they provide in general a suboptimal solution (local minimum). However, in practice, they are viable design methods for several problems.

VIII.3 SYNTHESIS OF ROBUST CONTROLS BY PARAMETRIC OPTIMIZATION

In this section we will review the problem of calculating the control by parametric optimization in order to develop an algorithm for calculating an optimal and robust control, without structural stress, i.e. that guarantees a prespecified degree of stability of the system (in Anderson's and Moore's sense (AND-71) and make the performance index insensitive to small variations (around nominal values) of system parameters.

VIII.3.1. Optimal Control With Prespecified Degree of Stability

Let us consider a controllable and observable system:

$$\mathring{X}(t) = A\ X(t) + B\ U(t)$$
$$Y(t) = C\ X(t)$$

and the traditional quadratic criterion: /166

$$\int_{t_o}^{\infty} (X^T\ Q\ X + U^T\ R\ U)dt \qquad (8\text{-}3\text{-}2)$$

Until now we were interested in calculating a decentralized control stabilizing the system, i.e. that places the poles of the system in the left half of the complex plan. However from a practical viewpoint, it is desirable to calculate a gain which places the poles in a specified area often considered as in figures (8.3a) and (8.3b).

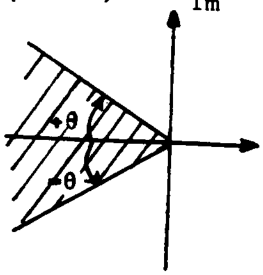

Fig. 8.3a

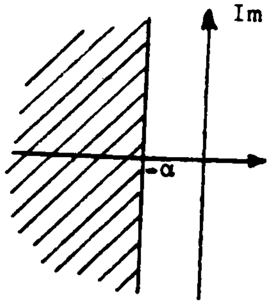

Fig. 8.3b

In this section, we want to achieve a pole placement in the area of figure (8.3b) and therefore to assure a prespecified degree of stability α with a closed loop system. Anderson and Moore (AND-71) show that it suffices to replace the quadratic criterion (8-3-2) by:

$$J(K) = \int_{t_o}^{\infty} e^{2\alpha t}\ (X^TQ\ X + U^T\ R\ U)\ dt \qquad \alpha > 0$$

The problem of optimization therefore becomes:

$$\min_K J(K) = \int_{t_o}^{\infty} e^{2\alpha t}\ (X^T\ Q\ X + U^T\ R\ U)\ dt$$

$$\text{sub} \quad \mathring{X} = AX + BU \qquad (8\text{-}3\text{-}3)$$
$$Y = CX$$
$$U = -\ KY = -\ KCX$$

To solve this problem let's change variables as follows:

$$\overline{X} = e^{\alpha t} X$$
$$\overline{U} = e^{\alpha t} X$$
$$\overline{Y} = e^{\alpha t} Y$$

which leads to the following problem: /167

$$\min_K J(K) = \int_{t_o}^{\infty} (\overline{X}^T Q \overline{X} + \overline{U}^T R \overline{U}) \, dt$$

$$\text{sub } \mathring{\overline{X}} = \overline{A}\overline{X} + B\overline{U} \qquad \text{with } \quad \overline{A} = A + \alpha I$$
$$\overline{Y} = C\overline{X}$$
$$\overline{U} = -K\overline{Y} = -KC\overline{X}$$

where again

$$\min_K J(K) = \int_{t_o}^{\infty} (\overline{X}^T Q \overline{X} + \overline{U}^T R \overline{U}) \, dt$$

$$\text{sub } \mathring{X} = (\overline{A} - BKC) \overline{X} = \overline{D}\overline{X} \qquad (8\text{-}3\text{-}4)$$
$$\text{with } \overline{D} = A + \alpha I - BKC$$

The existence of a solution to this problem is guaranteed if the pair (A,B) is controllable and the pair $(A, Q^{1/2})$ is observable; in effect Anderson and Moore (AND-71) show that if $(\overline{A},B)$ is controllable and $(\overline{A}, Q^{1/2})$ observable, then the pairs (A,B) and $(A,Q^{1/2}$ are also.

The solution of (8-3-4) guarantees the asymptotic stability of the closed loop system $\dot{\overline{X}} = \overline{D}\,\overline{X}$; we have seen that this solution is given by:

$$\frac{\partial \overline{J}}{\partial K} = 2\,(RKC - B^T \overline{P})\, \overline{S}\, C^T$$
$$f(K) = \overline{D}^T \overline{P} + \overline{P}\,\overline{D} + Q_1(K) = 0 \qquad (8\text{-}3\text{-}5)$$
$$g(K,\overline{X}_o) = \overline{D}\,\overline{S} + \overline{S}\,\overline{D}^T + \overline{X}_o = 0$$
$$\text{with } \quad \overline{X}_o = E\{\overline{X}(o) \;\; \overline{X}^T(o)\}$$

Note that if $\alpha = 0$ then $\overline{A} = A$ and $\overline{D} = D$ and we return to the stabilization problem already posed.

It is easy to see that the solution to problem (8-3-3) is the same as the solution to problem (8-3-4) because U: = - K Y, U = - KY, and that this solution guarantees the asymptotic stability of the original system with degree of stabililty α, and that the minimum values of criteria J and J are identical.

In short, to guarantee that the poles of the closed loop system are placed to the left of the right $-\alpha$ of the complex plan (fig. 8.3b), simply apply the calculated control by minimizing the conventional quadratic criterion (equations 8-2-7) after replacing the dynamic matrix A by $A + \alpha I$.

Note that if it is decided that the control should belong to the class of controls under any structural stress K_C, the solution to problem (8-3-3) is obtained by applying Geromel's and Bernussou's algorithm to equations (8-3-5). Let us recall that to guarantee the existence of the solution, the control structure /168
must permit system pole placement in the shaded area of figure (8.3b).

VIII.3.2. Optimal Control Minimizing the Sensitivity of the Performance Index (TGAR-85b)

We have seen that to calculate an optimal control, it is necessary to have exact knowledge of the system parameters. However the parameters of the system are never exactly known, but are given by nominal values which generally may vary or may be uncertain. Thus the optimal controls calculated by accounting for nominal parameter values do not remain optimal during small parameter variations. It is therefore important to calculate a robust control for these variations, by trying to minimize the sensitivity of the performance index for parameter variations. To measure this sensitivity, we adopt the following approach (YAH-77).

Let q be a scalar parameter of the system; if q changes into $q+\Delta q$ the criterion may be approximated by:

$$J(q+\Delta q) = J(q) + \Delta J(q) = J(q) + \Delta q.\, \frac{dJ}{dq}$$

Since the variation of the criterion is proportional to the gradient of the criterion relative to parameter q, we shall measure the sensitivity of the criterion by the modulus of the gradient of the criterion with respect to the system parameters.

Let us consider the problem already posed in the preceding sections by assuming that the system parameters are uncertain, i.e.

$$\min J_1(A,B,C,K) = \int_{t_o}^{\infty} (X^T Q X + U^T R U)\, dt$$

$$\text{sub } \overset{\circ}{X} = (A-BKC)\, X = DX$$

This problem may be expressed in the form:

$$\min J_1 (A,B,C,K,P) = Tr(P X_o)$$

$$\text{sub } f(A,B,C,K) = D^T P + P D + Q_1(C,K) = 0$$

$$\text{with } Q_1(C,K) = Q + C^T K^T R K C$$

Let us apply theorem (8-2-1) to calculate the derivatives of J_1 relative to its arguments, we have:

$$L_1(A,B,C,K,P,S) = Tr(P X_o) + Tr\left[S^T f(A,B,C,K)\right] \qquad (8\text{-}3\text{-}6)$$

The stationarity conditions give: /169

$$\frac{\partial L_1}{\partial A} = \frac{\partial J_1}{\partial A} = J_A = 2\ PS$$

$$\frac{\partial L_1}{\partial B} = \frac{\partial J_1}{\partial B} = J_B = -\ 2\ PS\ C^T K^T$$

$$\frac{\partial L_1}{\partial C} = \frac{\partial J_1}{\partial C} = J_C = 2\ K^T (RKC-B^TP)\ S \qquad (8\text{-}3\text{-}7)$$

$$\frac{\partial L_1}{\partial C} = \frac{\partial J_1}{\partial K} = J_K = 2\ (RKC - B^TP)\ S\ C^T$$

with

$$\frac{\partial L_1}{\partial S} = f(A,B,C,K,P) = D^T P + PD + Q_1(C,K) = 0$$

$$\frac{\partial L_1}{\partial P} = g(A,B,C,K,S) = DS + SD + X_o = 0$$

Since the sensitivity measurements (gradients of criterion) are calculated analytically, this allows us to define a new performance index (criterion) J_2:

$$J_2 = J_1 + \frac{1}{4}\, Tr\, (J_A^T.L.J_A) + \frac{1}{4}\, Tr\, (J_B^T.M.J_B) + \frac{1}{4}\, Tr\, (J_C.E.J_C^T) + \frac{1}{4}\, Tr\, (J_K.F.J_K^T)$$

where L, M, E, F are the positive semi-defined symmetrical matrices of the appropriate dimensions.

At this level, a question arises: Is it necessary to include all gradients J_A, FJ_B, FJ_C and J_K to minimize the sensitivity of criterion J_1? The answer is NO, because according to (8-3-7):

$$\begin{aligned} J_B &= - J_A C^T K^T \\ J_C &= K^T J_{KC} \quad , \quad K_{KC} = 2(RKC - B^T P)\, S \\ J_K &= J_{KC} C^T \end{aligned}$$

Note that under optimum conditions we have $J_A = 0 (J_A < \varepsilon_A)$ which means that $J_B = 0$ $(J_B < \varepsilon_B)$. Also it suffices to have $J_{KC} = 0$ $(J_{KC} < \varepsilon_{KC})$ to have $J_{KC} = 0$ $(J_K < \varepsilon_C)$ and $J_K = 0$ $(J_K < \varepsilon_K)$. Thus it suffices to minimize measurements J_A and J_{KC}.

To come to this result, let us assume that A and B vary simultaneously, along with K and C. In this case, we have according to (8-3-6) (to obtain these results using the variations method: see appendix 4):

$$J_{AB} = \frac{\partial L_1}{\partial (A,B)} = \frac{\partial L_1}{\partial A - \partial B.KC} = J_A = 2\ PS$$ /170

$$J_{KC} = \frac{\partial L_1}{\partial (KC)} = 2\ (RKC - B^T P)\ S$$

J_{AB} represents the sensitivity of the criterion with respect to variations of A and B and J_{KC}, the sensitivity of the criterion with respect to variations of K and C. We may therefore define a new performance index J_3:

$$J_3 = J_1 + \frac{1}{4} \operatorname{Tr} (J_{AB}^T.L.J_{AB}) + \frac{1}{4} \operatorname{Tr} (J_{KC}.F.J_{KC}^T) \qquad (8\text{-}3\text{-}8)$$

where L and F are two positive semi-defined symmetrical matrices, of the appropriate dimensions. The problem now consists of determing the gain K which minimizes J_3 and satisfies the stresses on the structure:

$$\min_{K \in K_C} J_3(K) = Tr\,(P\,X_o) + Tr\,(SPLPS) + Tr\left[(RKC-B^TP)\,SFS\,(RKC-B^TP)^T\right]$$

$$\begin{aligned}
\text{sub} \quad f(K) &= D^TP + PD + Q_1(K) = 0 \\
g(K,X_o) &= DS + S\,D^T + X_o = 0 \qquad (8\text{-}3\text{-}9) \\
\text{with} \quad D &= A - BKC \\
Q_1(K) &= Q + C^TK^T\,RKC
\end{aligned}$$

with A,B,C given by their nominal values.

To find the required optimality conditions, let us write the Lagrangian expression:

$$L_3 = J_3 + Tr\,(V^Tf) + Tr\,(W^Tg)$$

from which we obtain the stationarity conditions:

$$\frac{\partial L_3}{\partial K} = \frac{\partial J_3}{\partial K} = 2\,\left[(RKC-B^TP)V + R(RKC-B^TP)\,SFS-B^TWS\right]\,C^T \qquad (8\text{-}3\text{-}10a)$$

$$\frac{\partial L_3}{\partial V} = f(K,P) = D^TP + P\,D + Q_1(K) = 0 \qquad (8\text{-}3\text{-}10b)$$

$$\frac{\partial L_3}{\partial W} = g(K,S) = DS + S\,D^T + X_o = 0 \qquad (8\text{-}3\text{-}10c)$$

$$\frac{\partial L_3}{\partial S} = f_1(K,P,S,W) = D^TW + WD + Q_2(K,P,S) = 0 \qquad (8\text{-}3\text{-}10d)$$

$$\frac{\partial L_3}{\partial P} = g_1(K,P,S,V) = DV + V\,D^T + Q_3(K,P,S,X_o) = 0 \qquad (8\text{-}3\text{-}10e)$$

with

$$\begin{aligned}
D &= A-BKC \\
Q_1(K) &= Q + C^T\,K^T\,RKC \\
Q_2(K,P,S) &= SPLP + PLPS + (RKC-B^TP)^T(RKC-B^TP)SF + \\
&\quad + FS(RKC-B^TP)^T\,(RKC-B^TP) \\
Q_3(K,P,S,X_o) &= X_o+SSPL+LPSS-B(RKC-B^TP)SFS-SFS(RKC-B^TP)^TB^T
\end{aligned}$$

In the case of a state feedback control (C = I and F = 0), the stationarity conditions (8-3-10) are simplified in: /171

$$\frac{\partial J_3}{\partial K} = (RK - B^T P)\ V - B^T W\ S$$

with

$$\begin{aligned} f &= D^T P + PD + Q + K^T RK = 0 \\ g &= D\ S + S\ D^T + X_o = 0 \\ f_1 &= D^T W + W\ D + SPLP + PLPS = 0 \\ g_1 &= D\ V + V\ D^T + X_o + SSPL + LPSS = 0 \end{aligned} \qquad (8\text{-}3\text{-}11)$$

and the optimal control is given by:

$$\frac{\partial J_3}{\partial K} = 0 \quad \rightarrow \quad \begin{aligned} K^* &= R^{-1} B^T P + R^{-1} B^T W\ S\ V^{-1} \\ &= K_1^* + K_2^* \end{aligned} \qquad (8\text{-}3\text{-}12)$$

where K^*_1 is the optimal gain already found with respect to the nominal values of the system parameters (corresponding to the solution of min J_1)(and K^*_2 the gain part that minimizes the sensitivities of the performance index J_1. Note that this control depends on the initial conditions and that the independence of the gain is rediscovered only with respect to the initial conditions with $K^*_2 = 0$.

How can equations (8-3-10) be solved to obtain an optimal control?

If there is no structural stress,then equations (8-3-10) may be solved by using a gradient method: from an initial gain matrix stabilizing the system, four Lyapunov equations may be solved successively and in the following order: (8-3-10b) and (8-3-10c) then (8-3-10d) and (8-3-10e) then the gradient calculated and the gain reactualized by doing $K \leftarrow K - a \left.\frac{dJ}{dK}\right|$ until convergence ($a > 0$: no progression).

If a structural stress is applied to the control, we suggest using a projected gradient method and in this case the steps of the method are those of Geromel's and Bernussou's algorithm, namely:

Step 1: Initialization of algorithm by a gain K^1

stabilizing the closed loop system and satisfying the structural stress.

Step 2: Calculate matricial gradient equation (5-3-10), i.e.:

solve $f\ (K^i, P^i) = 0 \rightarrow P^i$

solve $g\ K^i, S^i) = 0 \rightarrow S^i$

solve $f_1\ (K^i, P^i, S^i, W^i) = 0 \rightarrow W^i$ /172

solve $g_1\ (K^i, P^i, S^i, V^i) = 0 \rightarrow V^i$

then $\dfrac{\partial J_3}{\partial K} = J_{3K}\ (V^i, W^i, K^i, P^i, S^i)$

Step 3: Projection of gradient to stresses, then determination of feasible direction G^i: $G^i = \Pi_{K_C}\ (\dfrac{\partial J_3}{\partial K})$

where Π_{K_C} is the projection operator on K_C.

Step 4: Optimality test:

if $\|G^i\| < \varepsilon$ Stop

otherwise go to the next step.

Step 5: Look for an optimal progression step by solving the monovariable problem:

$$\min_{a > 0} J\ (K^i - a\ G^i)$$

I.e. $a = a^i$ the solution

Step 6: Calculate $K^{i+1} = K^i - a^i\ G^i$, and go to step 2.

Note that the initial gain may be obtained using Armentano's and Singh's algorithm (see VIII.2.2.). It is evident that step 2 requires more computer time than the corresponding step of the problem without minimizing the sensitivity of the criterion, as we have two more Lyapunov equations for each iteration to solve: this is the price of the robust control. The efficiency of the approach essentially depends on the algorithm used for solving the Lyapunov equations. Let us mention in this regard Hoskin et al's (HOS-77) and Kleinman et al's (KLE-68) algorithms.

The projection step remains the same in section VIII.2.2.b., and consists of cancelling the terms corresponding to zero K_{ij}.

In order for the problem to keep a meaning for each iteration, the gain obtained for each iteration must satisfy the stresses and stabilize the system. The following lemma gives the conditions for this to be the case:

Lemma 8-3-1

Let $J(K) = K \in R^{mxp}$ be the matricial function defined by (8-3-8), differential relative to K, $\forall K$ such that stable (A-BKC); therefore:

a) if $\forall K \in K_C, \exists a > 0$ such that $J(K-aG) < J(K)$ when $G \neq 0$, where G is the projection of gradient $\frac{dJ}{dK}$ over the stress set represented by K_C.

b) If $(A, Q^{1/2})$ is an observable pair and matrix X_0 is defined as positive, then the algorithm gives a stabilizing control for each iteration.

Demonstration /173

This lemma was given by Geromel and Bernussou (GER-79a) to demonstrate the stability of his algorithm; it is also valid in our case and it may be demonstrated as follows:

a) By linearizing around a, we obtain (see GER-79c):

$$J(K-aG) = J(K) + a \left.\frac{dJ(K-aG)}{da}\right|_{a=0} \qquad (8\text{-}3\text{-}13)$$

using Kleinman's lemma (KLE-66) (see appendix 4), we may express:

$$\left.\frac{dJ(K-aG)}{da}\right|_{a=0} = - \mathrm{Tr}\,\{ (\frac{dJ}{dK})^T \, . \, G \} \qquad (8\text{-}3\text{-}14)$$

Let us replace (8-3-14) in (8-3-13), we obtain:

$$J(K-aG) = J(K) - a \ Tr \{ (\frac{dJ}{dK})^T . G \}$$

Since $G = \Pi_{K_C}(\frac{dJ}{dK})$, where Π_{K_C} is the projection operator, then:

$$J(K-aG) - J(K) = a \ TR \ (G^T G)$$

from which for $G \neq = 0$, there is an $a > 0$ such that $J(KaG) < J(K)$

b) Like $K \in K^C$, we have $(K-aG) \in K_C$ thus it suffices to show that $J(K)$ is finite and consequently $J(K-aG) \ J(K) < \infty$

Let us express J_3 in the form:

$$J_3 = Tr(P \breve{X}_o) + Tr(SPLPS) + Tr \ (RKC-B^TP) \ SFS(RKC-B^TP)^T$$
$$= J_{31} \quad + \quad J_{32} \quad + \quad J_{33}$$

If $(A,Q^{\frac{1}{2}})$ is observable and D asymptotically stable, then P is defined positive and with finite elements, i.e. $0 < P < \infty$.

If X_0 is defined positive and D asymptotically stable, then S is also defined positive and with finite elements, i.e. $0 < S < \infty$. We therefore have:

$$\left.\begin{matrix} 0 < P < \infty \\ 0 < X_o < \infty \end{matrix}\right\} \rightarrow J_{31} = Tr(P \ X_o) < \infty$$

$$\left.\begin{matrix} 0 < P < \infty \\ 0 < S < \infty \\ 0 < L < \infty \end{matrix}\right\} \rightarrow 0 < PS < \infty \quad \rightarrow \quad J_{32} = Tr \ (SPLPS) < \infty$$

$$\left.\begin{matrix} R > 0 \\ K,C,B \text{ with finite elements} \end{matrix}\right\} \rightarrow (RKC-B^TP) \ S < \infty \quad \rightarrow \quad J_{33} < \infty$$

from which $J_3 = J_{31} + J_{32} + J_{33} < \infty$ /174
with a) we have $J(K-aG) < J:(K) < \infty$

The gain verified for each iteration must therefore:

i) $(K-aG) \in K_C$ or $(P(K-aG) > 0$

ii) $J(K-aG) \leq J(K)$

Finally, with the same procedure for adapting the iteration step, the algorithm flow chart is that of figure 8.2 while adding two more Lyapunov equations.

Example 8-3-1

The algorithm proposed was used on a double precision IBM 370/165, for example (8 -2-1) and for the same initial data, namely: $a^1 = 0.1$, $\pi = 1.2$; $\nu = 0.05$; $\varepsilon = 10^{-3}$, $L = I$, $X_o = I$, $F = 0$ and K^1 is that of (8-2-16).

Convergence was obtained in 32 iterations, the suboptimal gain being:

$$K^{32} = \left[\begin{array}{cc|cc|cc} 1.549 & 0.437 & & & & \\ \hline & & 1.526 & 0.57 & & \\ & & 0.993 & 2.33 & & \\ \hline & & & & 1.91 & 0.275 \end{array}\right]$$

The natural values of the closed loop system (with K^{32}) are:

$$\{-2.018423 \mp 1.460728,\ -4.44452 \mp 1.573168,\ -6.438916,\ -5.187874\}$$

The initial value of the criterion was $J(K^1) = 8.935$ and at convergence $J_3(K^{32}) = 3.232$ and $J_1(K^{39}) = 3.049$.

To compare with the results of example (8-2-1), note that the value of J_1 is almost equivalent to $J(K^{59})$ of example (8-2-1) and J_3 is 0.077 times higher than $J(KI^{59})$; the calculation time is twice as long as for example (8-2-1) (no parameter variations).

VIII.3.3. Minimizing the Sensitivity of the Performance Index with Prespecified Degree of Stability

In this section, we bring together the results of the two preceding sections, while determining a control minimizing the sensitivities of the performance index, guaranteeing a prespecified degree of stability according to Anderson's and Moore's definition (see VIII.3.1) and verifying the structural stress applied.

The optimization problem considered is therefore: /175

$$\underline{\text{Prob. 1}} : \min_{K \in K_C} J(K) = \int_{t_o}^{\infty} e^{2\alpha t} (X^T QX + U^T RU)\, dt$$

$$\text{under}: \quad \overset{o}{X} = AX + BU$$
$$Y = CX$$
$$U = -KY$$

According to section VIII.3.1, the solution to this problem is equivalent to the solution of:

$$\underline{\text{Prob. 2}} : \min_{K \in K_C} J(K) = \int_{t_o}^{\infty} (\bar{X}^T Q\bar{X} + \bar{U}^T R\bar{U})\, dt$$

$$\text{under}: \quad \overset{o}{\bar{X}} = \bar{A}\,\bar{X} + B\,\bar{U} \text{ with } \bar{A} = A + \alpha I$$
$$\bar{Y} = C\,\bar{X}$$
$$\bar{U} = -K\,\bar{Y}$$

If the system parameters are given by their exact value, then the solution to problem 2 is obtained by applying Geromel's and Bernussou's algorithm to equations (8-2-7) after converting A into $\bar{A}$. .

If the parameters are uncertain and are known by their nominal value, then the solution is obtained by applying the expanded algorithm proposed in the preceding section for equations (8-3-10) by changing A to A.

If $\alpha = 0$, we return to the problems of the preceding sections.

VIII.4 SYNTHESIS BY EXPANSION AND CONTRACTION

In the preceding sections, we presented a few methods for synthesizing decentralized controls while assuming that the system could be decomposed into disjoined subsystems without information sharing between subsystems. However in reality we find (traffic regulation (ATH-67, ISA-73A), economical systems (AOK-76), power systems (CAL-78, SIL-78)) as many systems do not have this property, i.e. systems where information is shared between subsystems.

The control of these systems may be calculated using expansion and contraction techniques. The idea of this technique is to expand (under certain conditions) the state space of the system to have a space with a larger dimension (for equal limits), which contains any information of the original space and which reveals a decomposition into disjoined subsystems. Traditional optimization techniques may therefore be used to calculate a decentralized control in this expanded space. Then the expanded control is contracted to return to the control used on the original system.

VIII.4.1. Expansion and Contraction of the Problem /176

Let us consider system S defined by:

$$S : \dot{X} = A\ X + B\ U \qquad X(o) = X_o \tag{8-4-1}$$

with $X \in R^n$ and $U \in R^m$. The following criterion is associated with this system:

$$J(X_o,\ U) = \int_o^\infty (X^T\ Q\ X + U^T\ R\ U)\ dt$$

with $Q \geq 0$ and $R > 0$.

Let us associate the pair (S, J), with $(\bar{S}, \bar{J})$ where system $\bar{S}$ and criterion $\bar{J}$ are defined by:

$$\bar{S} : \dot{\bar{X}} = \bar{A}\ \bar{X} + \bar{B}\ U \qquad \bar{X}(o) = \bar{X}_o$$

$$\bar{J}(\bar{X}_o, U) = \int_o^\infty (\bar{X}^T\ \bar{Q}\ \bar{X} + U^T\ \bar{R}\ U)\ dt$$

with $\bar{Q} \geq 0$, $\bar{R} > 0$. $\bar{X} \in R^{\bar{n}}$ & $\bar{n} \geq n$

I.e. the linear transformation:

$$\bar{X} = T X \qquad (8\text{-}4\text{-}2)$$

where T is a matrix of dimension $\bar{n} \times n$ of full rank. Note by $X(t; X_0, U)$ and by $\bar{X}(t; \bar{X}_o, U)$ the states of systems S and $\bar{S}$ for the initial conditions X_0 and $\bar{X}_o$ and the fixed input U. Ikeda et al (IKE-81) use transformation T to connect the pairs (S,J) and $(\bar{S},\bar{J})$ in the following inclusion context.

Definition 8-4-1 (IKE-81f)

The pair $(\bar{S},\bar{J})$ includes the pair (S,J) if a matrix J exists such that, $\forall X_o$ of S, the choice:

$$\bar{X}_o = T X_o \qquad (8\text{-}4\text{-}3)$$

of the initial state of $\bar{S}$ implies that:

$$X(t; X_o, U) = T^I \bar{X}(t; X_o, U) \qquad \forall t > 0$$
$$J(X_o, U) = \bar{J}(X_o, U)$$

$\forall$ the fixed input U (T^I generalized inverse of T).

If the pair $(\bar{S},\bar{J})$ includes the pair (S,J) then $(\bar{S},\bar{J})$ is said to be an EXPANSION of (S,J) and (S,J) is a CONTRACTION of $(\bar{S},\bar{J})$. Note that the optimization problem associated with (S,J) is equivalent to that associated with (S,J) if (8-4-3) is verified.

If transformation (8-4-2) is applied, then the matrices of the expanded system and of the original system are connected by:

$$\bar{A} = T A T^I + M, \quad \bar{B} = T B + N$$
$$\bar{Q} = (T^I)^T Q T^I + M_Q \quad \& \quad \bar{R} = R + N_R$$

where M, N, M_Q and N_R are constant matrices observing /177 the conditions below:

Theorem 8-4-1 (IKE-81)

The pair $(\overline{S},\overline{J})$ includes the pair (S,J) if

$$\text{a) } MT = 0,\ N = 0,\ T^I M_{Q_1} T = 0 \text{ and } N_R = 0$$

or
$$\text{b) } T^I M^i T = 0,\quad T M^{i-1} N = 0,\quad M_Q M^{i-1} T = 0$$
$$M_Q M^{i-1} N = 0 \text{ and } N_R = 0 \qquad i = 1,2,\dots,\overline{n}$$

Under what conditions is control $U = -\overline{K}\,\overline{X}$ "contractible" into $U: = -KY$?

Definition 8-4-2 (IKE-81)

The control law $U = -\overline{KX}$ of expansion $\overline{S}$ is contractible into $U = -KY$ to control the original system S. if $\overline{X}_0 = T X_0$ implies $K\,X(t;\,X_0,U) = \overline{K}\,\overline{X}(t;\,\overline{X}_0,\,U)\ \forall t \geq 0$ and for any fixed input U.

It is easy to see that if $MT = 0$ and $N = 0$, then control $-\overline{K}\,\overline{X}$ is contractible into $-K X$, with $K = \overline{K}\,T$.

Let us recall that the reason for this development is that the control is calculated for the expanded system $\overline{S}$ using a standard optimization technique. This control is then contracted to apply it to the original system S.

VIII.4.2. Overlapping Decomposition

Given system S of (8-4-1), let us assume that it is of order n and that its state is decomposed into three components X_1, X_2 and X_3 of dimension n_1, n_2 and n_3 respectively:

$$X = (X_1^T\ X_2^T\ X_3^T)^T$$
$$\text{and } n = n_1 + n_2 + n_3$$

let us assume that the system has two control stations, i.e.:

$$U = (U_1^T\ U_2^T)^T$$

with $U_1 \in R^{m_1}$, $U_2 \in R^{m_2}$ & $m = m_1 + m_2$. The system may be described by:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \qquad (8\text{-}4\text{-}5)$$

Let us decompose the state into two overlapping components:

$$\bar{X}_1 = (X_1^T \; X_2^T)^T$$

$$\bar{X}_2 = (X_2^T \; X_3^T)^T$$

then the expanded state vector is: $\bar{X} = (\bar{X}_1^T \; \bar{X}_2^T)^T$ and it is related /178
to X by:

$$\bar{X} = T\,X$$

T being the rectangular matrix of dimension $\bar{n} \times n$ with

$$\bar{n} = n_1 + n_2 + n_3$$

and defined by:

$$T = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

where I_1, I_2 and I_3 are unity matrices of the appropriate dimensions.

We have seen that this transformation defines an expression $\bar{S}$:

$$\bar{S} : \dot{\bar{X}} = \bar{A}\,\bar{X} + \bar{B}\,U$$

where:

$$\bar{A} = T\,A\,T^I + M \qquad \bar{B} = T\,B + N$$

Note here that the possibilities in T, M, N are not unique (IKE-81), Ikeda et al (IKE-81) retain the following selections:

$$T^I = \begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & \tfrac{1}{2}I_2 & \tfrac{1}{2}I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & \tfrac{1}{2}A_{12} & -\tfrac{1}{2}A_{12} & 0 \\ 0 & \tfrac{1}{2}A_{22} & -\tfrac{1}{2}A_{22} & 0 \\ 0 & -\tfrac{1}{2}A_{22} & \tfrac{1}{2}A_{22} & 0 \\ 0 & -\tfrac{1}{2}A_{32} & \tfrac{1}{2}A_{32} & 0 \end{bmatrix} \quad \text{and} \quad N = 0$$

Therefore the expanded system S becomes:

$$\bar{S}: \begin{bmatrix} \dot{\bar{X}}_1 \\ \hdashline \dot{\bar{X}}_2 \end{bmatrix} = \left[\begin{array}{cc:cc} A_{11} & A_{12} & 0 & A_{13} \\ A_{21} & A_{22} & 0 & A_{23} \\ \hdashline A_{21} & 0 & A_{22} & A_{23} \\ A_{31} & 0 & A_{32} & A_{33} \end{array}\right] \begin{bmatrix} \bar{X}_1 \\ \hdashline \bar{X}_2 \end{bmatrix} + \left[\begin{array}{c:c} B_{11} & B_{12} \\ B_{21} & B_{22} \\ \hdashline B_{21} & B_{22} \\ B_{31} & B_{32} \end{array}\right] \begin{bmatrix} U_1 \\ \hdashline U_2 \end{bmatrix}$$

By comparing systems S and $\bar{S}$, it is clear that the decomposition of S is disjoined and a standard optimization technique may be used to calculate the control of each subsystem separately.

Expansion $\bar{S}$ may be represented as two interconnected systems

$$\bar{S}_1 : \mathring{\bar{X}}_1 = \bar{A}_1 \bar{X}_1 + \bar{B}_1 U_1 + \bar{A}_{12} \bar{X}_2 + \bar{B}_{12} U_2$$

$$\bar{S}_2 : \mathring{\bar{X}}_2 = \bar{A}_2 \bar{X}_2 + \bar{B}_2 U_2 + \bar{A}_{21} \bar{X}_1 + \bar{B}_{21} U_1$$

where

$$\bar{A}_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$$

$$\bar{A}_2 = \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \ \& \ \bar{B}_2 = \begin{bmatrix} B_{22} \\ B_{32} \end{bmatrix}$$

are matrices of the coupled subsystems:

$$\bar{S}_1^D : \mathring{\bar{X}}_1 = \bar{A}_1 \bar{X}_1 + \bar{B}_1 U_1$$

$$S_1^D : \mathring{\bar{X}}_2 = \bar{A}_2 \bar{X}_2 + \bar{B}_2 U_2$$

and

$$\bar{A}_{12} = \begin{bmatrix} 0 & A_{13} \\ 0 & A_{23} \end{bmatrix}, \quad \bar{A}_{21} = \begin{bmatrix} A_{21} & 0 \\ A_{31} & 0 \end{bmatrix}, \quad \bar{B}_{12} = \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} \ \& \ \bar{B}_{21} = \begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix}$$

are interconnection matrices between subystems. Let us associate the following criteria with the decoupled subsystems:

$$\bar{J}_1 (\bar{X}_{10}, U_1) = \int_0^\infty (\bar{X}_1^T \bar{Q}_1 \bar{X}_1 + U_1^T \bar{R}_1 U_1)\, dt$$

$$\bar{J}_2 (\bar{X}_{20}, U_2) = \int_0^\infty (\bar{X}_2^T \bar{Q}_2 \bar{X}_2 + U_2^T \bar{R}_2 U_2)dt$$

where $\bar{X}_{10}$, $\bar{X}_{20}$ are initial states of $\bar{S}_1^D$ & $\bar{S}_2^D$ & $\bar{Q}_1$, $\bar{Q}_2$, $\bar{R}_1$ & $\bar{R}_2$ of the matrices of appropriate dimensions. The global criterion is:

$$\bar{J}(\bar{X}_o, U) = \int_o^{\infty} (\bar{X}^T \bar{Q} \bar{X} + U^T \bar{R} U)\, dt$$

with
$$\bar{Q} = \mathrm{diag}(\bar{Q}_1, \bar{Q}_2)$$
$$\bar{R} = \mathrm{Diag}(\bar{R}_1, \bar{R}_2)$$

according to part a) of theorem (8-4-1), it is concluded that $\bar{J}(\bar{X}_o, U)$ is an expansion of $J(X_0, U)$:

$$J(X_o, U) = \int_o^{\infty} (X^T Q X + U^T R U)\, dt$$

with
$$Q = T^T Q T \qquad R = \bar{R}$$

$J(X_o, U)$: criterion associated with the original.

At this level, the decentralized control of the expanded /180
system is calculated, i.e.:

$$U_1 = - \bar{K}_1 \bar{X}_1$$
$$U_2 = - \bar{K}_2 \bar{X}_2$$

so as to optimize the decoupled subsystems $\bar{S}_1^D$ and $\bar{S}_2^D$ relative to criteria $\bar{J}_1 (\bar{X}_{10}, U_1)$ and $\bar{J}_2 (\bar{X}_{20}, U)$. The overall control is the contraction of K, given by:

$$\bar{K} = \left[\begin{array}{c|c} \bar{K}_1 & 0 \\ \hline 0 & \bar{K}_2 \end{array}\right] = \left[\begin{array}{cc|cc} \bar{K}_{11} & \bar{K}_{12} & 0 & 0 \\ \hline 0 & 0 & \bar{K}_{23} & \bar{K}_{24} \end{array}\right]$$

and the control to be installed on the original system is the contraction of K, given by.

$$K = \bar{K}\, T = \begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} & 0 \\ 0 & \bar{K}_{23} & \bar{K}_{24} \end{bmatrix}$$

The calculation of the suboptimality index of the solution may be found in (IKE-81) (SIL-82a).

Recently Ikeda and Siljak (IKE-84) expanded the inclusion principle, used here, for a unified treatment of input, output and sate expansion and contraction. For more details refer to (IKE-84).

VIII.5. OTHER SYNTHESIS METHODS

Other methods of synthesizing decentralized controls are found in literature, in addition to those presented here:

-Hierarchized Computer Methods:

.Hassan's and Singh's (HAS-78b) algorithm with 3 calculation levels,

.Hassan's, Singh's and Titli's (HAS-79) algorithm with 3 calculation levels,with prespecified degree of stability,

.Xinoglas', Mahmoud's and Singh's (XIN-82) algorithm with 2 calculation levels.

-Computer Methods Using an Interconnection Model:

.Use of a general interconnection model (HAS-78a),

.Use of a follower model (HAS-80, CHE-81).

In the text below, we will discuss in detail only Xinoglas', Mahmoud's and Singh's algorithm, the most recent of the algorithms given.

VIII.5.1. Algorithm With Two Calculation Levels by Xinoglas, Mahmoud and Singh

Xinoglas, Mahmoud and Singh (XIN-82) treated the problem of determining a decentralized control by state feedback $U = - K X$ /181 by minimizing the traditional quadratic criterion:

$$J = \frac{1}{2} \int_{o}^{\infty} (X^T Q X + U^T R U) \, dt \qquad X(o)$$

and satisfying the structural stress $K \in K_S$ such that:

$K_S = \{K \,/\, K = \text{diagonal block and stable } (A-BK)\}$

We have seen that the associated optimization problem is expressed:

$$\min_{K \in K_S} \; J = \mathrm{Tr}\{(Q + K^T R K)\, S\}$$

$$\text{under} \quad g(S, X_o) = S(A-BK)^T + (A-BK)S + X_o = 0$$

$$\text{with} \quad X_p = E\{X(o)\, X^T(o)\} = \mathrm{diag}\,(X_i)$$

Let us express:

$$\left.\begin{aligned} A_d &= \mathrm{diag}\{A\} \\ A_o &= A - A_d \end{aligned}\right|$$

then the problem is written:

$$\min_{K \in K_S} \; J = \mathrm{Tr}\{(Q + K^T R K)\, S\}$$

$$\text{under} \quad \begin{aligned} g(S,X_o) &= S(A_d-BK)^T + (A_d-BK)S + Z + X_o = 0 \\ Z &= A_o S + S A_o^T \end{aligned}$$

The Langrangian associated with this problem is:

$$\mathbf{L} = \mathrm{Tr}\;\{(Q + K^T RK)\; S\} + \mathrm{Tr}\,\{P\; g(S,X_o)\} + \mathrm{Tr}\;\{T(A_o S - S A_o^T - Z)\}$$

the optimality conditions give:

$$\frac{\partial \mathbf{L}}{\partial T} = 0 \quad \rightarrow \quad Z = A_o P + P A_o^T$$

$$\frac{\partial \mathbf{L}}{\partial Z} = 0 \quad \rightarrow \quad T = P$$

$$\frac{\partial \mathbf{L}}{\partial P} = 0 \quad \rightarrow \quad (A_d-BK)S + S(A_d-BK)^T + X_o + Z = 0 \qquad (8\text{-}5\text{-}1)$$

$$\frac{\partial \mathbf{L}}{\partial S} = 0 \quad \rightarrow \quad (A_d-BK)^T T + T(A_d-BK) + Q + K^T RK + A_o^T P + P A_o = 0 \qquad (8\text{-}(\text{-}2)$$

$$\frac{\partial \mathbf{L}}{\partial K} = 0 \quad \rightarrow \quad K = R^{-1} B^T M_d S_d^{-1}$$

$$M_d = \mathrm{diag}\{TS\}$$

$$S_d = \mathrm{diag}\{S\}$$

To solve these optimality conditions Xinoglas, Mahmoud and Singh (XIN-82) propose the following two level algorithm:

Step 1: Arbitnarily select an initial decentralized gain K^m.

Step 2: If $(A_d - B K^m)$ is stable, go to step 3. /182
otherwise use Armentano's and Singh's algorithm (see VIII.2.2) to calculate $K^m \in K_d$.

Step 3: Begin the two level calculation algorithm by selecting 2^m and T^m. Send them with K^m to the 1st level. m=1.

Step 4: At the 1st level, solve equations (8-5-1) and (8-5-2) using Bartel's and Stewart's technique (BAR-72), send S^m and T^m to the second level.

Step 5: Calculate new predicitons of Z, P and K as follows:

$$Z^{m+1} = A_o S^m + S^m A_o^T$$
$$P^{m+1} = T^m$$
$$K^{m+1} = R^{-1} B^T M_d^m (S_d^m)^{-1}$$

If the conditions:

$$\left.\begin{array}{l} \|Z^{m+1} - Z^m\| < \varepsilon_Z \\ \|P^{m+1} - P^m\| < \varepsilon_P \\ \|K^{m+1} - K^m\| < \varepsilon_K \end{array}\right|$$

are satisfied, then K^{m+1} is the optimal solution searched for, otherwise calculate Z^{m+1}, p^{m+1} and K^{m+1}

$$\left.\begin{array}{l} Z^{m+1} = c_1 Z^m + d_1 Z^{m+1} \\ P^{m+1} = c_2 P^m + d_2 P^{m+1} \\ K^{m+1} = c_3 K^P + d_3 K^{m+1} \end{array}\right|$$

where the constants c_j and d_j satisfy $c_j + d_m = 1$ for j=1,2, and 3.
ε_Z, ε_P, ε_K are small positive constants (precision).
Go to step 2.

Remarks

1 - The calculation is divided into two levels. At the 1st level, two Lyapunov equations are sovled and, at the 2nd level, a prediction routine is used - numerical correction (SIN-78a). The convergence of thealgorithm amy be analyzed as with the two level optimization algorithm (SIN-78a).

2 - The algorithm proposed requires less computer time than Geromel's and Bernussou's algorithms (GER-79a) and Chens, Mahmouds' and Singh's (CHE-84) algorithms. For example the authors installed the algorithm on PDP-10 and treated example (8-2-1), the initial control being:

$$K^1 = \left[\begin{array}{cc|cc|cc} 0,9004 & 0,2487 & & & & \\ \hline & & 1,143 & 0,4902 & & \\ & & 0,6536 & 2,219 & & \\ \hline & & & & 1,184 & 0,32 \end{array}\right]$$

convergence was obtained in 9 iterations for the following optimal gain: /183

$$K^9 = \left[\begin{array}{cc|cc|cc} 1.191 & 0.375 & & & & \\ \hline & & 1.601 & 0.5386 & & \\ & & 0.9343 & 2.273 & & \\ \hline & & & & 1.613 & 0.3583 \end{array}\right]$$

with the value of criterion $J(K^9) = 2.718$. The CPU time was:

initial block diagonalization 2.56 sec
suboptimal block diagnalization 7.2 sec

(see remark 4 of section VIII.2.3)

VIII.6 CONCLUSION

In this chapter, we essentially presented methods for synthesizing decentralized (static*) controls of linear systems in the absence of fixed modes. We focussed particularly on parametric optimization techniques from which we proposed an algorithm

*For dynamic control synthesis, see appendix 5.

for calculating a robust control with variations of the system parameters and assuring a prespecified degree of stability for a closed loop system.

CHAPTER IX - DECENTRALIZATION OF THE CONTROL OF THE WATER-STEAM CYCLE OF A SHIP /185

IX.1 INTRODUCTION

The purpose of this chapter is to apply the results developed in the previous chapters to the model of a water-steam cycle of a ship ("Chevalier Valbelle" of the "Chargeurs Réunuis" Company).

We briefly describe the boiler by presenting a mathematical model, patterned after the identification made by Piasco and Diep (PIA-79), from which we develop the synthesis of the control of this system.

IX.2 DESCRIPTION AND MATHEMATICAL MODEL OF THE PROCESS /186

The fundamental diagram of the water-steam cycle is depicted in figure (9.1). The steam produced in the balloon is overheated before being sent into the turbines (high and low pressure), which may develop a power of 38,000 HP to propel the ship at 24 knots. After expansion in the turbines, the steam is condensated in the capacitor and the condensated water is reheated then is sent into the balloon by a turbo-pump.

The part of the cycle we are interested in here is the steam generator unit which is made up of a balloon and two overheaters, subjected to the action of the furnace: the balloon receives the supply of water, and produces saturated steam, which is then dried through two overheaters. The combustion is regulated by an air-fuel mixture (load signal) which supplies the burners.

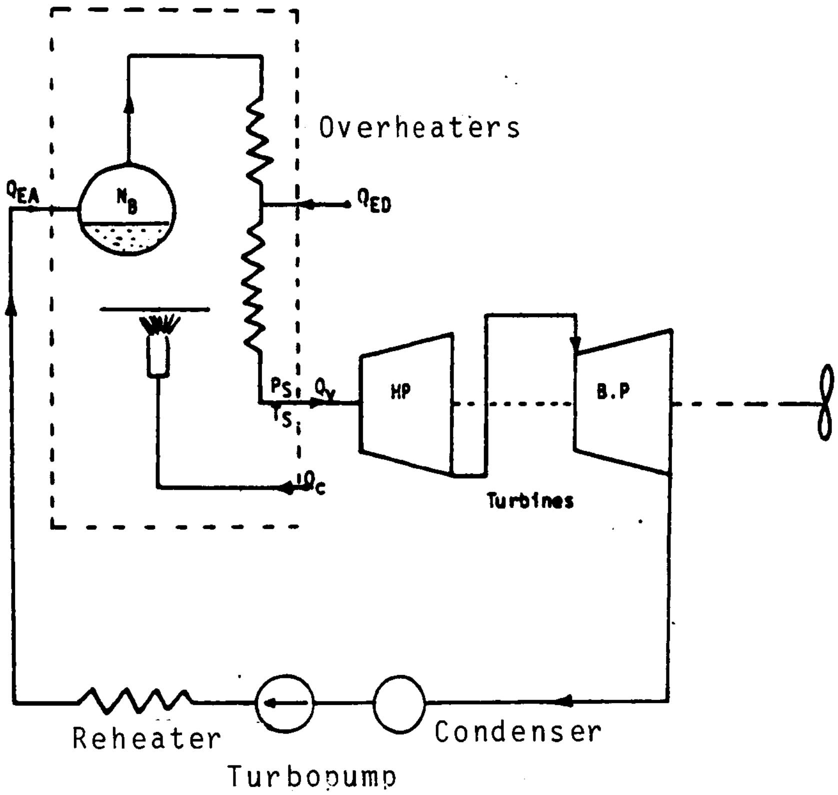

Fig. 9.1 - Water-Steam Cycle

The boiler characteristics are:

.vaporization rate: 58/65 tons/hour
.overheated steam pressure : 62 bars
.overheated steam temperature: 513°C /187
.Fuel rate: 7/9 tons/hour

The boiler has 4 input magnitudes (3 controls and one perturbation) which are:

Q_C	load signal (fuel and air rate)	controls
Q_{EA}	feed water rate	controls
Q_{ED}	cooled overheated water flow rate	controls
Q_V	steam flow rate at turbine intake	perturbations

and three outlets which are:

P_S Steam pressure at overheater outlet

T_S Steam temperature at overheater outlet

N_B Water level of balloon

IF we consider that the intakes are valve opening controls U_{EA}, U_C and U_{ED} which control the flow rates Q_{EA}, Q_C and not the flow rates themselves, then the diagram of figure (9.2) shows the main elements of the system sutided and the different intake and outlet magnitudes.

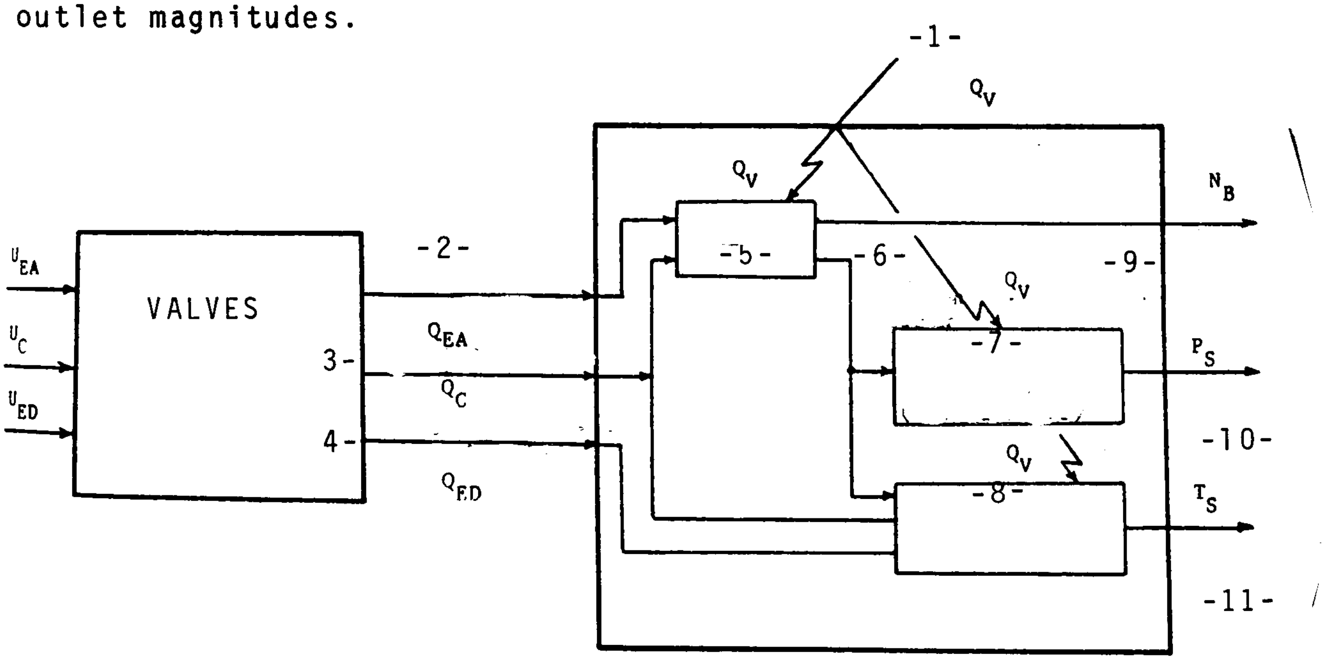

Fig. 9.2 - Physical Model

Key: 1-Steam flow rate Q_V; 2-Feed water flow rate Q_{EA};
3-Load signal Q_C; 4-Cooled overheated water flow rate Q_{ED}
5-Balloon;
6-Balloon pressure; 7-Load loss (overheater);
8-Overheaters; 9-Balloon level;
10-Steam pressure P_S; 11-Steam temperature.

The transfer diagram of the installation is given by Piasco and Diep (PIA-79) (fig. 9.3). This figure shows that the perturbational inlet Q_V (steam flow rate) alone affects the two blocks of the diagram (dotted line) and therefore the modes corresponding to these blocks are uncontrollable in a decentralized /188

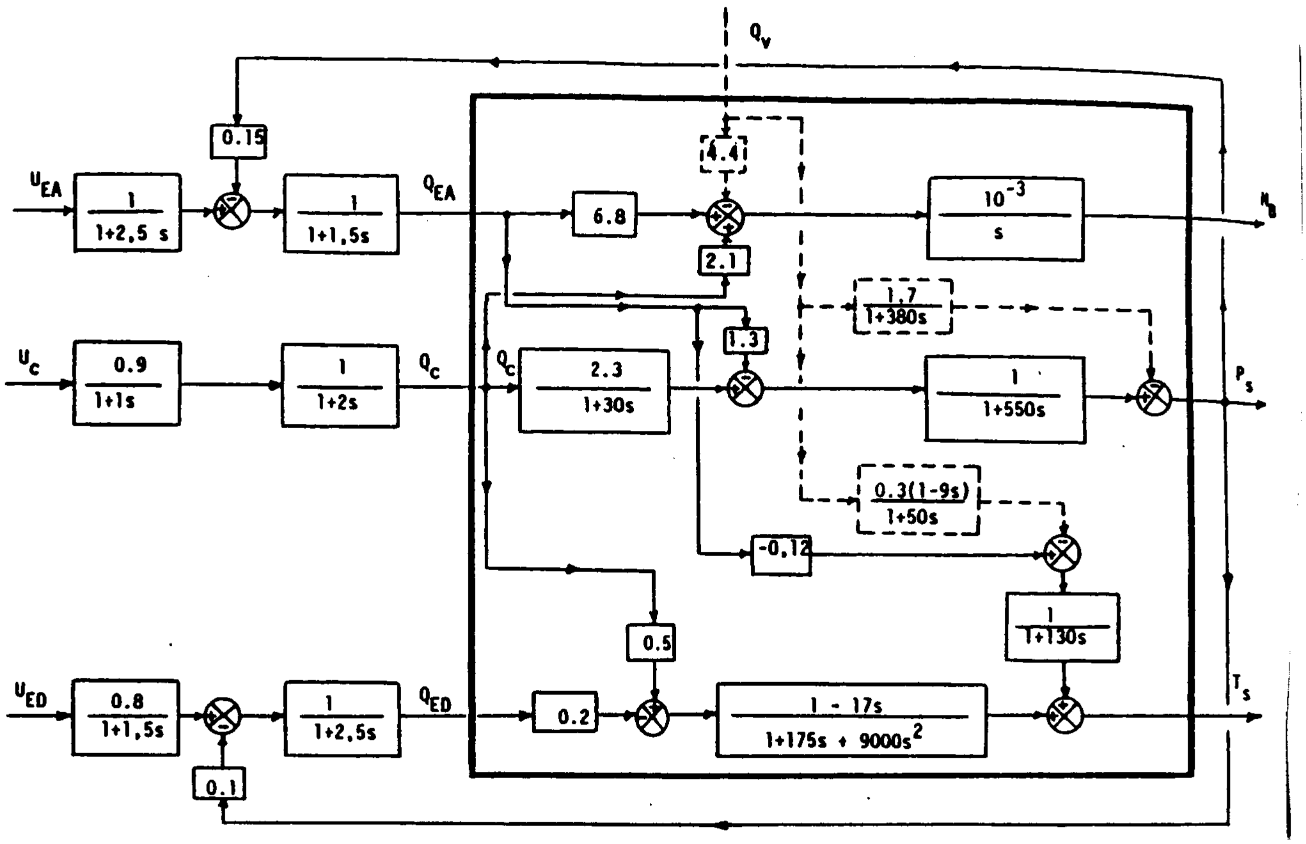

Fig. 9.3 - Transfer Diagram of the Model

manner. Consequently, they cannot be controlled by an output feedback, regardless of the nature of this feedback. This allows us to simplify the block diagram of fig. (9.3) by eliminating the blocks corresponding to these uncontrollable modes to consider only the controllable and observable part of the system. We thus obtain the block diagram of fig. (9.4).

The state representation of the system of figure (9.4) is given in plate (9.1), and the modes of the open loop system are:

IX,3 DECENTRALIZED CONTROL BY DYNAMIC OUTPUT FEEDBACK /189

IX.3.1. Characterization

The process to regulate has a control structure in which we may distinguish three main interconnected loops:

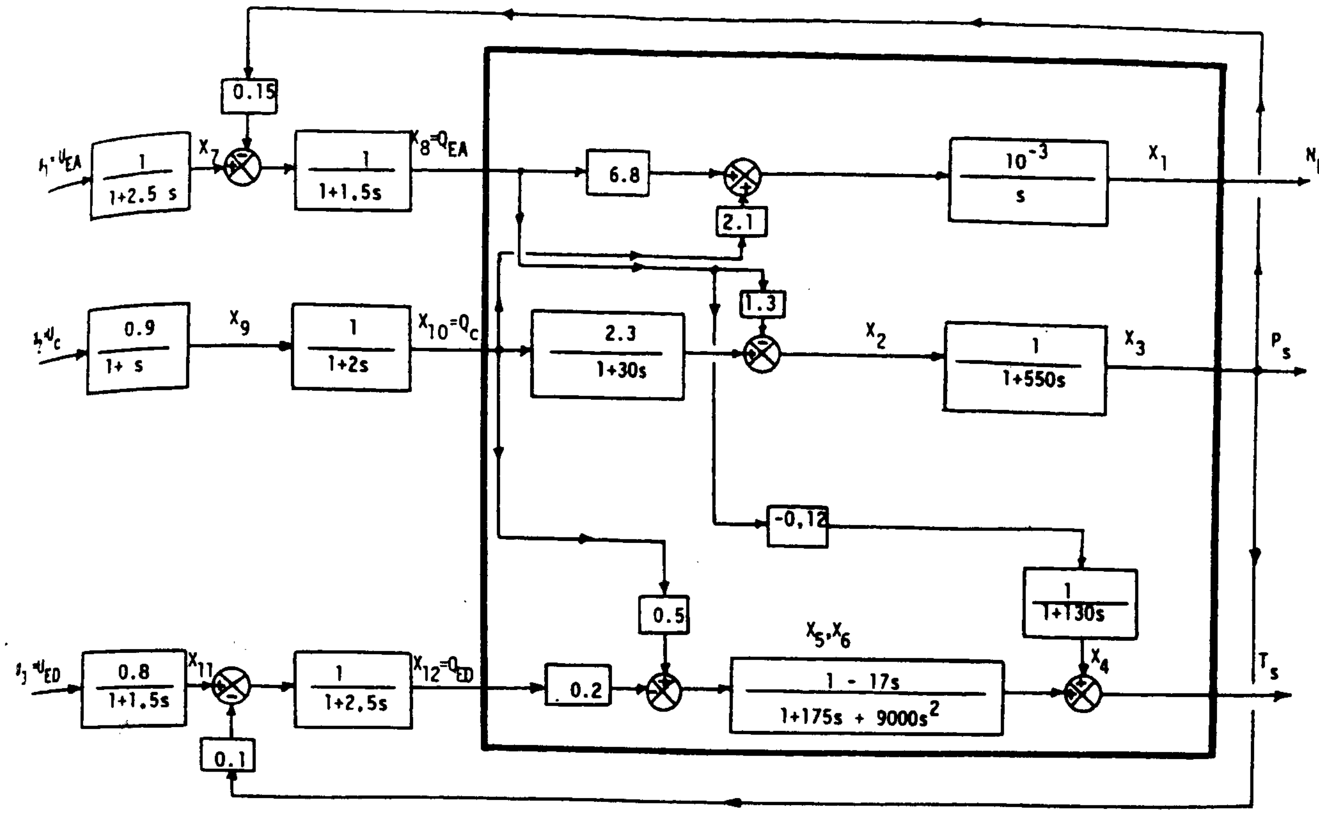

Fig. 9.4 - Transfer Diagram of Mode (Without Perturbation)

-The balloon N_B level, controlled by the feed water rate A_{EA} (or U_{EA}) and subjected to the influence of the load signal Q_C.

-The temperature at the outlet of the 2nd overheater T_S, controlled by the water flow rate of cooled overheated water Q_{ED} (or U_{ED}) and influenced by the load sigal Q_C (or U_C) and by the feed water rate Q_{EA} (or U_{EA}).

-The steam pressure at the collector P_S, controlled by the load signal Q_C (or U_C) and affected by the feed water flow Q_{EA} (or U_{EA}).

To control these loops, we select an output feedback control with a completely decentralized structure, i.e.:

$\{0, -0{,}00181, -0{,}00769, -0{,}00972 \mp 0{,}00405j, 0{,}03333,$
$-0{,}4, -0{,}4, -0{,}5, -0{,}6666, -0{,}6666, -1\}$

MATRIX A OF SYSTEM

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0068	0.0	0.0021	0.0	0.0
0.0	-0.0333	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0767	0.0	0.0
0.0	0.0018	-0.0018	0.0	0.0	0.0	0.0	-0.0024	0.0	0.0	0.0	0.0
0.0	0.0	0.0	-0.0077	0.0	0.0	0.0	-0.0009	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	-0.0195	-0.0110	0.0	0.0	0.0	0.0056	0.0	-0.0022
0.0	0.0	0.0	0.0	0.0100	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	-0.4000	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-0.100	0.0	0.0	0.0	0.6667	-0.6667	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0000	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5000	-0.5000	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.6670	0.0
0.0	0.0	-0.0400	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.4000	-0.4000

MATRIX B OF SYSTEM

0.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.4000	0.0	0.0
0.0	0.0	0.0
0.0	0.9000	0.0
0.0	0.0	0.0
0.0	0.0	0.8
0.0	0.0	0.0

MATRIX C OF SYSTEM

1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	1.0000	-0.1700	1.0000	0.0	0.0	0.0	0.0	0.0	0.0

Plate 9.4. - State Representation of System of Fig. 9.4

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$$U = \begin{bmatrix} U_{EA} \\ U_C \\ U_{ED} \end{bmatrix} = \begin{bmatrix} k_{11} & & \\ & k_{22} & \\ & & k_{33} \end{bmatrix} \begin{bmatrix} N_B \\ P_S \end{bmatrix} = K_d \ Y$$

To affirm the existence of a decentralized control of form (9-3-1), we will test for the existence of fixed modes, with respect to this structure, using two methods:

IX.3.1.a. Sensitivity Algebraic Test

Application of algorithm (4-5-1) to the system shows that s = - 0.00769 est a type i structural decentralized fixed mode, the sensitivity matrix of this mode with respect to the control being:

$$SK\ (s=-0{,}00769) = \begin{bmatrix} -\ 0.8278 \times 10^{-17} & -\ 0{,}1104 \times 10^{-17} & 0{,}24 \times 10^{-6} \\ 0{,}3852 \times 20^{-17} & 0{,}5002 \times 10^{-18} & -0{,}1 \times 10^{-6} \\ -\ 0{,}5435 \times 10^{-31} & -\ 0{,}7246 \times 10^{-32} & 0{,}1576 \times 10^{-20} \end{bmatrix} \quad (9\text{-}3\text{-}2)$$

This matrix shows that the mode in question is sensitive (for an accuracy of 10^{-6}) only to elements k_{13} and k_{23}.

IX.3.1b. Graphic Test

The directed graph associated with a closed loop system (described by the state equation of plate (9.1) and a decentralized control K_d) is given by fig. 9.5. Application of theorem (5-4-1) shows that the system ahs a typei structural decentralized fixed mode corresponding to the sate peak x_4, because this peak is not contained in a highly connected component of the graph.

The system therefore has a stable type i structural fixed mode in s = - 0.00769, and a decentralized dynamic control may stabilize the system, but a free pole placement is impossible /192

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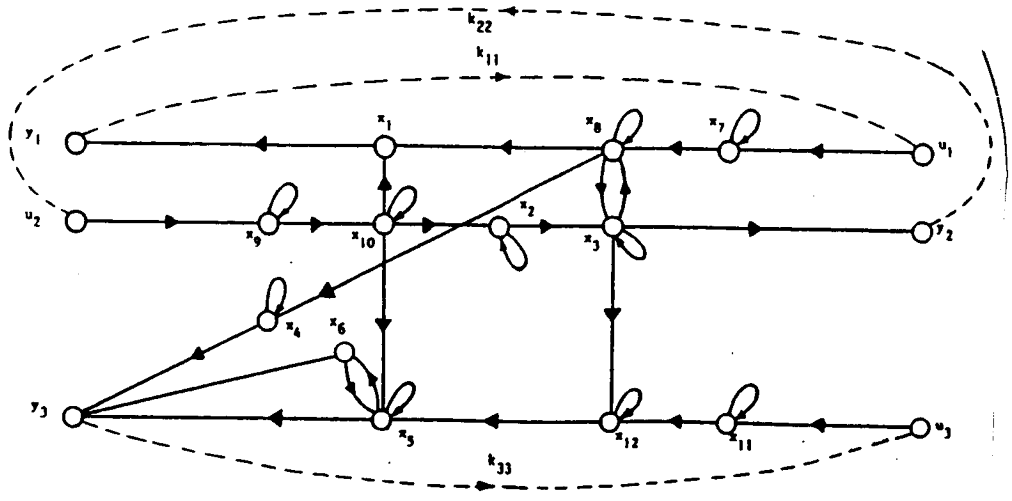


Fig. 9.5 - Graph Associated With the System

with this structure. Since the fixed mode is located very close to the imaginary axis of the complex plan, we consider it to be insufficiently stable. Therefore the problem is to find a structure making it possible to avoid this fixed mode, i.e. permitting a free pole placement of the system.

IX.3.2. Control

IX.3.2a. Stress Relief: Looking For Supplementary Links

According to the sensitivity matrix SK of (-3-2), the mode s = - 0.00769 is sensitive to feedback loops k_{13} and k_{23}. Assuming that the costs associated with these two loops are equal, we have two possible structures:

$$K_1 = \begin{bmatrix} k_{11} & & k_{13} \\ & k_{22} & \\ & & k_{23} \end{bmatrix} \quad \text{and} \quad K_2 = \begin{bmatrix} k_{11} & & \\ & k_{22} & k_{23} \\ & & k_{33} \end{bmatrix}$$

Remark:

This result may also be obtained according to the system graph by determining the feedback loops which form a highly connected component containing peak x_4. Note that these solutions are not optimal, because an optimal solution contains max (m,p) = 3 feedback loops.

IX.3.2.2b. Stress Relief: Selecting the Structure

Let us apply the procedure proposed in section (VII.3.3b.) to this example. The arc sets associated with the state peaks of the system are (TAR-85c):

$$
\begin{array}{ll}
K_{x_1} = \{k_{11},k_{21}\} & K_{x_7} = \{k_{11},k_{12},k_{13}\} \\
K_{x_2} = \{k_{21},k_{22},k_{23}\} & K_{x_8} = \{k_{11},k_{12},k_{13},k_{21},k_{22},k_{23}\} \\
K_{x_3} = \{k_{11},k_{12},k_{13},k_{21},k_{22},k_{23}\} & K_{x_9} = \{k_{21},k_{22},k_{23}\} \\
K_{x_4} = \{k_{13},k_{23}\} & K_{x_{10}} = \{k_{21},k_{22},k_{23}\} \\
K_{x_5} = \{k_{23},k_{33},k_{13}\} & K_{x_{11}} = \{k_{33}\} \\
K_{x_6} = \{k_{23},k_{33},k_{13}\} & K_{x_{12}} \ \{k_{13},k_{23},k_{33}\}
\end{array}
$$

note that:

$$
\begin{array}{ll}
K_{x_1} \subset K_{x_8} = K_{x_3} & K_{x_4} \subset K_{x_{12}} \\
K_{x_2} = K_{x_9} = K_{x_{10}} \subset K_{x_8} = K_{x_3} & K_{x_7} \subset K_{x_8} = K_{x_3} \\
K_{x_{11}} \subset K_{x_5} = K_{x_6} = K_{x_{12}} &
\end{array}
$$

therefore the arc sets to consider are only: /193

$$K_{x_1},\ K_{x_2},\ K_{x_4},\ K_{x_7}\ \&\ K_{x_{11}}$$

but $K_{x_{11}} \cap K_{x_i} = \emptyset \quad i=1,2,4,11.$

k_{33} is necessary. The problem is therefore:

Find K* such that $\mathrm{Card}\,(K^* \cap K_{x_i}) = 1 \quad i=1,2,4,7$

i.e.

$$Z = K_{x_1} \cup K_{x_2} \cup K_{x_4} \cup K_{x_7} = \{k_{11},k_{21},k_{22},k_{23},k_{12},k_{13}\}$$
$$= \{z_1,z_2,z_3,z_4,z_5,z_6\}$$

Matrix L is (see VIII.3.2b.)

$$L = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The solution to this problem is given by sovling the following linear program (assuming that the costs associated with feedback loops are equal):

$$\min\ (w_1 + w_2 + w_3 + w_4 + w_5 + w_6)$$

$$\text{under}\quad \begin{aligned} & w_1 + w_2 \geqslant 1 \\ & w_2 + w_3 + w_4 \geqslant 1 \\ & w_1 + w_5 + w_6 \geqslant 1 \\ & w_4 + w_6 \geqslant 1 \end{aligned} \qquad\qquad w_i^* = \begin{cases} 1 \\ 0 \end{cases} \qquad i=1,\dots,6$$

the resolution of this problem gives two solutions:

$$W_1^* = (1\ 0\ 0\ 1\ 0\ 0)^T \quad \text{and} \quad W_2^* = (0\ 1\ 0\ 0\ 0\ 1)$$

with a minimum cost equal to 2. Let us add to these solutions the feedback loop k_{33}, we obtain the two following structures:

$$K_3 = \begin{bmatrix} k_{11} & & \\ & & k_{23} \\ & & k_{33} \end{bmatrix} \quad \text{and} \quad K_4 = \begin{bmatrix} & & k_{13} \\ k_{21} & & \\ & & k_{33} \end{bmatrix}$$

these structures are optimal because they contain only three feedback loops.

The control structures (K_1, K_2, K_3 and K_4) thus assure the absence of fixed modes. Consequently, the stabilization and free pole placement of the system, using a dynamic control with one of these structures, are possible.

The synthesis of decentralized dynamic controls (decentralized observers) is not treated in our paper. However we show in appen- /194
dix 5 that this problem is reduced to the synthesis of decentralized static controls of an augmented system. Accordingly, the synthesis of a dynamic control will not be discussed in this example.

IV.4 DECENTRALIZED CONTROL BY STATIC OUTPUT FEEDBACK

According to the results of section VII.4, free pole placement of the system by static feedback is feasible only if the graph associated with a closed loop system contains 12 circuits of dimension 1,2,3,....,12, each of these circuits containing an arc associated with a different feedback loop. Consequently, a free pole placement of the system is not feasible even with a centralized output feedback *as defined above) as $n = 12 > m x p = 3x3$. However, decentralized stabilization by the structure:

$$K = \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & k_{23} \\ 0 & 0 & k_{33} \end{bmatrix}$$

is structurally feasible, as the necessary condition of the absence of fixed modes is verified. Additionally, the modes to be stabilized of the system are those associated with states x_1, x_5 and x_6 which may be influenced by feedback loops k_{11} (for x_1) and k_{23} and k_{33} (for x_5 and x_0).

Minimization of the criterion:

$$\int_0^{\infty} (y^T y + u^T u)\, dt$$

suing Geromel's and Bernussou's algorithm (see VIII.2) for initial data:

$$a^1 = 0{,}1, \quad \pi = 1{,}2, \quad \nu = 0{,}01, \quad \varepsilon = 10^{-3}, \quad X_o = I$$

$$K^{(1)} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

is achieved after 40 iterations, the optimal gain matrix being:

$$K^{(40)} = \begin{bmatrix} -1{,}014 & 0 & 0 \\ 0 & -0{,}5578 & -0{,}1199 \\ 0 & 0 & 0{,}07796 \end{bmatrix}$$

The initial value of the criterion is $J(K^{(1)}) = 1087.6$ and has the convergence $J(K^{(40)}) = 567.37$.

The natural values of the closed loop system (with $K^{(40)}$) are:
$\{-0.999813, -0{,}67691, -0.666941, -0.503396, -0.383232, -0.400105, -0{,}030752, -0{,}009649 \mp 0{,}005297, -0.00518 \mp 0{,}001301, -0.007476\}$

To return to the problem of pole placement of the system, we have seen that the only way to accomplish this is to relief structural stresses. According to the graph associated with the system (fig. 9.5) we see that to have a circuit of dimension 1 (containing an arc associated with a feedback loop) there must be an arc associated with the state feedback loop x_7 or x_9 or x_{11} which is not accessible for measurements. Therefore it is necessary to consider a dynamic control to achieve free pole placement. However placement of the 11 poles of the system is feasible only using other outputs of the system as shown below. /195

The system has two time scales, its modes are therefore decomposed into two groups: fast (modes associated with valves) and slow (modes associated with the process). If the system outputs are redefined by outputs of the slow part (same outputs as before notated y_1, y_2 and y_3) and outputs of the fast part (i.e. outputs of valves which are accessible for measurements and representing flow rates Q_{EA}, Q_C and Q_{ED}) then the output vector Y^* ($Y^* \in R^{p^*}$, $p^* = 6$) becomes:

$$Y^* = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \hline y_4 \\ y_5 \\ y_6 \end{bmatrix} \begin{bmatrix} N_B \\ P_S \\ T_S \\ \hline Q_{EA} \\ Q_C \\ Q_{ED} \end{bmatrix} \left[\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & & & & & & \\ 0 & 0 & 1 & 0 & 0 & 0 & & & \mathbb{O} & & & \\ 0 & 0 & 0 & 1 & -0{,}17 & 1 & & & & & & \\ \hline & & & & & & 0 & 1 & 0 & 0 & 0 & 0 \\ & & \mathbb{O} & & & & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & & & & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x_1 \\ \vdots \\ x_6 \\ \hline x_7 \\ \vdots \\ x_{12} \end{bmatrix}$$

The feedback matrix is therefore of dimension m x p* = 3 x 6. Let us consider a decentralized control matrix (decentralization of two scales as follows):

$$K_d^* = \begin{bmatrix} k_{11} & & & k_{14} & & \\ & k_{22} & & & k_{25} & \\ & & k_{33} & & & k_{35} \end{bmatrix}$$

According to the system graph (fig. 9.5) the system has a type i structural fixed mode (that associated with the state peak x_4), and to eliminate it, one of the elements k_{13} or k_{23} must be eliminated. The two following structures are therefore obtained:

$$K_5 = \begin{bmatrix} k_{11} & & k_{13} & k_{14} & & \\ & k_{22} & & & k_{25} & \\ & & k_{33} & & & k_{36} \end{bmatrix} \quad K_6 = \begin{bmatrix} k_{11} & & & k_{14} & & \\ & k_{22} & k_{23} & & k_{25} & \\ & & k_{33} & & & k_{36} \end{bmatrix}$$

which permit free pole placement by dynamic control and stabilization by static control.

Minimization of: $\int_0^{\infty} (y^{*T} y^{*} + u^T u)\, dt$ /196

using Geromel's and Bernussou's algorithm, for the initial data:

$$a^1 = 0{,}1, \quad \pi = 1{,}2, \quad \nu = 0{,}01 \quad \varepsilon = 10^{-3}, \quad X_o = I$$

$$K^{(1)} = \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & -0{,}2 & 0 & 0 \\ 0 & -1 & 0{,}5 & 0 & -0{,}5 & 0 \\ 0 & 0 & -0{,}8 & 0 & 0 & -0{,}7 \end{array}\right]$$

is achieved after 147 iterations, the optimal gain matrix being:

$$K^{(147)} = \left[\begin{array}{ccc|ccc} -1{,}69 & 0 & 0 & -1{,}462 & 0 & 0 \\ & -0{,}4953 & -0{,}0992 & 0 & -0{,}3526 & 0 \\ 0 & 0 & 0{,}0481 & 0 & 0 & -0{,}4578 \end{array}\right]$$

The initial value of the criterion is $J(K^{(1)}) = 1023.02$ and at convergence $K(K^{(147)}) = 653.93$.

The natural values in a closed loop (with $K^{(147)}$) are:

$$\{-0.531 \mp 0{,}6077,\ -0.75 \mp 0{,}3103,\ -0{,}5335 \mp 0{,}3587,\ -0.0316,$$
$$-0.00396 \mp 0{,}00109,\ -0{,}00765,\ -0{,}0097 \mp 0{,}004688\}$$

Remarks

1 - According to the system graph, the structure:

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & & \\ k_{21} & k_{22} & k_{23} & & k_{25} & \\ & k_{32} & k_{33} & & & k_{26} \end{bmatrix}$$

permits free placement of 11 system poles (in the graph associated with the closed loop system there are 11 release circuits 2, 3, 4, ...,12 containing arcs associated with different feedback loops. Minimization of the aforementioned criterion suing Geromel's and Bernussou's algorithm, for the same initial data above and for:

$$K^{(1)} = \left[\begin{array}{ccc|ccc} -2 & 0.5 & 0,6 & -0,2 & 0 & 0 \\ -0,5 & -1 & 0,3 & 0 & -0,5 & 0 \\ 0 & 0.3 & -0,6 & 0 & 0 & -0,7 \end{array}\right]$$

gives at convergence (after 169 iterations) the following optimal gain matrix:

$$K^{(169)} = \left[\begin{array}{ccc|ccc} -0,8683 & 0,3072 & 0,2948 & -0,3055 & 0 & 0 \\ -0,2809 & -0,4554 & -0,008897 & 0 & -0,3138 & 0 \\ 0 & 0,02536 & 0,04406 & 0 & 0 & -0,3101 \end{array}\right]$$

The value of the criterion is $J(K^{(1)}) = 916.69$ and at con- /197
vergence $J(K^{(169)}) = 629.41$. The natural closed loop values (with $k^{(169)}$) are:

$$\{-0,74988 \mp 0,280153,\ -0,530897 \mp 0,246674,\ -0,533509 \mp 0,285356,$$
$$-0,031696,\ -0,003437,\ -0,005077,\ -0,007722,\ -0,009727 \mp 0,004655\}$$

2 - Analysis of the system under study in this chapter as a system with two time scales is performed by Chemouil (CHE-78).

IX.5 CONCLUSION

It was not easy to find a model of the physical process establishing unstable fixed modes.

However the example considered here, even if it is not sufficient, as far as we are concerned, illustrates the possibility of applying our results to real situations.

Large systems are characterized by a large number of control and measurement variables and are generally distributed geographically. To apply them to a traditional centralized control would lead to a too sophisticated regulator or to a too complex communication network. This is the reason for a "DECENTRALIZED CONTROL" which is moreover favored by technological and economic progress in mini and microcomputers. These decentralization stresses reveal theoretical linear problems.

Among the problems for linear and time-invaraint linear dynamic systems, we focussed on the problem of pole stabilization and placement by output (or state) feedback when decentralization stresses are taken into consideration. The conditions for the existence of a solution to this problem are given by Wang and Davison (WAN-73b) in terms of FIXED MODES which are invariant modes of the system with respect to the control structure applied: decentral- /200
ized stabilization is impossible if the fixed modes are unstable, and their presence makes free pole placement impossible (ch. II).

Owing to this important result, we have selected to focus our study on the notion of fixed modes by asking ourselves the following questions:

*How can fixed modes be detected (or characterized)? What is their physical meaning?

*What can we do when they are present? How can they be avoided?

*How can decentralized controls be synthesized in the absence of fixed modes?

Our first objective was therefore to provide an overview of methods for characterizing and detecting fixed modes and that exist in literature (ch. III. IV and V). This allowed us to understand the nature of fixed modes. We provided two possible interpretations:

.in terms of transmission zeros of certain subsystems.
.based on the notions of controllability and observability.

Actually, the notion of decentralized fixed modes is an extension of the notion of uncontrollable and/or unobservable fixed modes, in a decentralized manner, which are decomposed into two types:

Structurally uncontrollable modes under structural stress (type i and ii_1 structural fixed modes) which are associated with the system structure.

Structurally controllable modes under structural stress (nonstructural and structural type ii_2 fixed modes) which are associated with perfect equalities of certain system parameters.

WE have tried to unify the different results obtained with respect to these two types of fixed modes, and have developed a method of characterization based on the notion of the sensitivity of antural values, making it possible to test their existence and determine their nature.

Then we considered the problem of pole stabilization or placement in the presence of fixed modes. Since the presence of fixed modes is associated with the structure of the system or with equalities between paramters, then a change in the nature of the control or of its structure may solve the problem.

We have demonstrated (ch. VI) that the use of UNSTATIONARY decentralzied feedback laws (quantification of the control, time-variant feedback laws, nonlinear control) may stabilize structurally controllable fixed modes in a decentralized manner, and showed /201
(ch. VII) that stress relief of structures is the most natural

way to eliminate fixed modes. In this regard, we added the following three procedures to methods that already exist in literature for determining optimal control structures (with respect to the number of feedback loops or their associated cost): one is based on the notion of mode sensitivity and the other two are based on notions of graph theory.

Moreover, since fixed modes are defined relative to the output or state feedback control, then the use of another control principle (vibrational control, anticipation control) may stabilize the sytem. In this regard we have shown (ch. VI) that a vibrational control may stabilize the system for any type of fixed mode, and demonstrated the conditions for the existence of decentralized vibrational feedback laws for stabilizing nonstructural fixed modes.

At this stage, we posed the problem of snthesizing decentralized (or quasi-decentralized) static controls in the absence of fixed modes. The answer to this problem is given in chapter VIII which presents a synthesis of most of the existing algorithms. We also considered the problem of synthesizing robust controls (resistant to system parameter variations), and proposed a synthesis algorithm by parametric optimization which uses the method of the projected gradient (initially developed by Geromel and Bernussou) and which makes it possible to obtain a robust decentralized control (minimization of the criterion and of its gradients with respect to system parameters).

Finally, in addition to the numerous small illustrative examples presented in the different chapters, we devoted chapter IX to the application of different methodologies developed in our paper on the model of a steam generator of a ship. We analyzed the system and proposed a quasi-decentralized stabilizing control, as the system ahs a type i decentralized structural fixed mode.

We believe we have thus made a large enough synthesis of the problems arising when decentralizing continuous linear dynamic systems, and the solutions which may be applied to them. Nevertheless, we regret not having enough time to solve the problem of the synthesis of dynamic controls (observers) under structural stress: this problem may be reduced to the synthesis of static controls of an augmented system (appendix 5), but remains the problem of reducing dimensions for the selection of a minimal observer.

Another important point is that of pole stabilization and /202
placement using static decentralized controls: this problem has not yet been totally solved, as if we presented the conditions for the existence of such a control, to our knowledge, there are no systematic methods in existence for determining the control structure and this point remains to be investigated.

Let us point out that in this paper we considered only continuous linear systems, but the problems associated with fixed modes arise also for decentralized discrete systems, for which nothing has been done.

Finally, let us note that many complex physical systems are by anture stochastic processes which removes us from the deterministic approach that we have retained and leads us to the vast area of decentralized stochastic controls.

ZEROS IN MULTIVARIABLE SYSTEMS

The following multivariable linear system is considered:

$$\overset{o}{X}(t) = A\ X(t0 + B\ U(t)$$
$$Y(t) = C\ X(t) + D\ U(t)$$

where $X \in R^n$, $U \in R^m$, $Y \in R^p$ are state, control and output vectors, respectively, A, B, C, D are constant matrices of the appropriate dimensions, $n \geq 1$, $m \geq 1$, $p \geq 1$ and max $(m,p) \leq n$. The polynomial matrix:

$$P(s) = \begin{bmatrix} sI-A & B \\ C & D \end{bmatrix}$$

is called "system matrix" (ROS-70). If r = rank P(s), then Smith's expression for P(s) is:

$$S(s) = \left[\begin{array}{c|c} \text{diag}\ (s_1, \ldots, s_r) & 0_{r,p-r} \\ \hline 0_{m-r,r} & 0_{m-r,p-r} \end{array}\right]$$

where the s_i are invariant polynomials of P(s) (s_1 divides s_{1+1}) given by (GAM-66):

$$s_i = \frac{M_j}{M_{j-1}} \qquad j=1,2,\ldots,r \qquad \text{with } M_0 = 1$$

and where M_j is p.g.c.d. of all minors of order j of P(s).

In the frequency range, the system is described by its transfer matrix:

$$G(s) = C(sI-A)^{-1}\ B + D = \frac{N(s)}{d(s)}$$

if r - rank G(s), Smith's - McMIllan's expression of G(s) is:

$$M(s) = \left[\begin{array}{c|c} \mathrm{diag}\left(\frac{\varepsilon_1}{\psi_1}, \ldots, \frac{\varepsilon_r}{\psi_r}\right) & 0_{r,p-r} \\ \hline 0_{m-r,r} & 0_{m-r,p-r} \end{array}\right]$$

where $\frac{\varepsilon_i}{\psi}$ is the i-th invariant polynomial of N(s) divided by the characteristics polynomial of system d(s). Note that $\varepsilon_{i+1}(\psi_{i+1})$ divides $\varepsilon_{i+1}(\psi_{i+1})$.

It seems that Rosenbrock (ROS-70) was the first to treat and define the zeros of mulivariable systems in an appropriate and understandable manner. He introduced several zero notions (invariant zero, decoupling zero,...) of multivariable systems. Currently, the following zero concepts are found in literature:

Z1: Zero Element (E.Z.):

The zero elements are all roots of numerators of elements $g_{ij}(s)$ of the transfer function G(s).

Without forgetting the role and significance of these zeros for monovariable systems, these zeros do not have a special meaning for multivariable systems.

Z2: Decoupling Zeros (Z.D.) (ROS-70)

These zeros are associated with the decoupled modes of the system, they are s values for which the rank of one of the matrices (sI-A B) and/or $\binom{sI-A}{C}$ is not complete.

These zeros are therefore uncontrollable and/or unobservable modes of the system, and do not appear in the transfer matrix (because of the presence of a pole-zero simplification). They are decomposed into input decoupling zeros (Z.d.e.) (uncontrollable modes), output decoupling zeros (z.d.s) (unobservable modes) and input-output decoupling zeros (z.d.e.s.) (uncontrollable and unobservable modes, we have: Z.D. $=\{z.d.e.\} \cup \{z.d.s.\} - \{z.d.e.s.\}$

Z3: Transmission Zeros (Z.T.) (ROS-70)

These zerosa are the set of roots of all numerators, expressed by Smoth - McMillan as G(s). In terms of G(s) minors, they are the set of p.g.c.d. of the numerators of all minors of G(s) order r after adjusting these minors to have d(s) as a common denominator.

The Z.T. appear as zeros of certain G(s) elements and as poles of other elements. Physically they are associated with the transmission properties between intputs and outputs (see MAC-76). Note that Rosenbrock called them "zeros of the transfer matrix".

Z4: Invariant Zeros (Z.I.)

These are the set of roots of invariant P(s) polynomials. In terms of P(s) minors, they are p.g.c.d. roots of all minors of a maximum P(s) order.

These zeros are transmission zeros plus a few decoupling zeros. /207

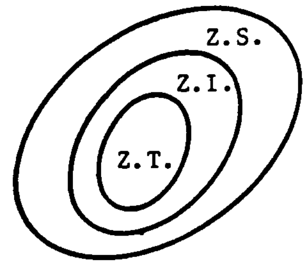
Z5: System Zeros (Z.S.) (ROS-74)

The system zeros are the set of p.g.c.d. roots of all P(s) minors in the form $P\,^{1,2,\ldots,n,n+i_1,\ldots,n+i_k}_{1,2,\ldots,n,n+j_1,\ldots,n+j_k}$ where the maximum nominal value of k is ℓ with $0 \leqslant \ell \leqslant \min(m,p)$.

These zeros are the Z.T. set plus the Z.D. set.

Z.S. = Z.T. $\cup$ Z.D.

Note that the Z.I. set is a Z.S. subset as shown in Vann's diagram to the right, and that if the system is controllable and observable, then the Z.S., Z.I. sets (defined from the P(s)) and Z.T. (defined from the G(s)) coincide because the Z.D. set is empty.



The zero sets above are all defined in terms of P(s) and G(s) minors, or in an equivalent manner in temrs of their invariant polynomials. However other definitions are found in literature in terms of frequency for which the rank of matrices P(s) and G(s) diminshes. These definitions are given below:

Z6: Wolovich (WOL-73a)

The zeros of the controllable and observable system (A,B,C,D) are z xomplexes which verify:

$$\text{rank } P(z) \leq \text{rank } P(s)$$

Z7: Wolovich (WOL-73b) **and Desoer & Schulman** (DES-74)

The transfer matrix may be factored out into G(s) = V(s) $T^{-1}(s)$ + D, where V(s) and T(s) are right prime polynomial matrices. Therefore the zeros of the system are z complexes which verify:

$$\text{Rank } V(z) < \text{rank } V(s)$$

It is evident (except for multiple zeros) that definitions Z6 and Z7 are equivalent to Rosenbrock's definition Z3.

Z8: Davison and Wang (DAV-74 and 76c)

The transmission zeros of system (Z,B,C,D) are complex z numbers which verify:

$$\text{rank } P(z) < n + \min(m,p),$$

in particular, the transmission zeros (including multiplicity) are roots of the p.g.c.d. of all minors of order n + min(m,p) of P(s).

Note that if rank P(s) < n = min(m,p) then all points of the complex plan are zeros of the system (degenerated system). It is clear that the present definition coincides with Rosenbrock's definition Z3. Note also that for the particular case of nongenerated systems with D = 0, the Z.T. defined here are the set of roots of the transmission polynomials of system (A,B,C) defined by Morse (MOR-73). /208

APPENDIX 2

STRUCTURAL RANK OF A MATRIX /209

Definition

A structure matrix $\overline{M}$ is a matrix which has a fixed number of zero elements in certain positions and arbitrary p elements in others. The space of R^p parameters is associated with nonzero elements such that each point $d \in R^p$ defines a matrix $M = \overline{M}(d)$. Conversely, for each matrix M with nonzero p elements there is a structured matrix $\overline{M}$ such that $M=\overline{M}(d)$ for $d \in R^p$. The structural rank of a matrix M, notated gr(M)* is given by:

$$gr(M) = \underset{d \in R^p}{\text{Max}} (\text{rank } \overline{M}(d)$$

Several algorithms are found in literature to determine the structural rank based on different notions, among which:

1 - Davison's algorithm (DAV-77), based on the fact that gr(M) is the rank of almost all matrices M(d) for $d \in R^p$. It

*gr: as "generic rank" in English

simply consists of arbitrarily selecting (pseudo-random generation of numbers) a point d and of determining the rank of M(d) by a standard technique.

2 - The algorithms based on an adaptation of graphic concepts (SHI-76, MOR-78, HOS-80, BUR-83, JOH-84,....).

3 - The algorithms based on the property that the structural rank of a matrix is the maximal rank determined as a function of its nonzero parameters. It is therefore the sum of nozero block diagonal dimensions of a maximjal pepmutation of matrix (EVA-84).

To possibly help the reader, let's go back to Evans and Kruser who give in (EVA-84) an APL computer program for the structural rank of a matrix.

APPENDIX 3 /210

ACCESSIBILITY MATRIX OF A DIRECTED GRAPH ASSOCIATED WITH A LINEAR SYSTEM

Given the linear system:

$$\overset{\circ}{X} = A\,X + B\,U \qquad X \in R^n,\ U \in R^m$$
$$Y = C\,X \qquad Y \in R^p$$

Let us associated the directed graph $D = (V,L) = (U \cup X \cup Y,\ L)$ with the system. The peaks V correspond respectively to inputs $U = \{u_1, \ldots, u_m\}$, to states $X = \{x_1, \ldots, x_n\}$ and to outputs $Y = \{y_1, \ldots, y_p\}$. L is the set of directed arcs (V_j, V_i) from peak V_j to peak V_i; the arcs (x_j, x_i), (u_m, x_i), $(x_j, y_i) \in L$ if and only if $a_{ij} \neq 0$, $b_{ij} \neq 0$ and $c_{ij} \neq 0$ respectively. The corresponding adjacency matrix is:

$$M = \begin{array}{cc} \begin{array}{ccc} X & U & Y \end{array} & \\ \begin{bmatrix} E_o & F_o & 0 \\ 0 & 0 & 0 \\ Q_o & 0 & 0 \end{bmatrix} & \begin{array}{c} X \\ U \\ Y \end{array} \end{array} \quad \text{with} \quad m_{ij} = \begin{cases} 1 & \text{if \& only if } (V_j, V_i) \in L \\ 0 & \text{otherwise} \end{cases}$$

Definitions

1 - Peak V_j is sais to be "accessible" from peak V_i if there is a path (drect or indirect) from V_i to V_j.

2 - Graph D is said to be "accessible from the input" if and only if the state peaks X are accessible from the input peaks U.

3 - Graph D is said to be an "accessible output" if and only if the output peaks are accessible from the state peaks.

4 - If graph D is accessible from the input and has an accessible output, then it is "accessible from the input-output".

The accessibility matrix corresponding to graph D is of dimension k x k (k=n+m+p) and in the form:

$$R = \begin{array}{cc} \begin{array}{ccc} X & U & Y \end{array} & \\ \begin{bmatrix} E & F & 0 \\ 0 & 0 & 0 \\ Q & H & 0 \end{bmatrix} & \begin{array}{c} X \\ U \\ Y \end{array} \end{array}$$

where E, F, Q and H are the state, input, output and input-output /211
matrices respectively, and are given by (SIL-78):

$$\bar{E} = E_o + E_o^2 + \dots + E_o^k$$
$$\bar{F} = (I + E_o + E_o^2 + \dots + E_o^{k-1})\ F_o$$
$$\bar{Q} = Q_o\ (I + E_o + E_o^2 + \dots + E_o^{k-1})$$
$$\bar{H} = Q_o\ (I + E_o + E_o^2 + \dots + E_o^{k-1})\ F_o$$

and $e_{ij} = 1$, $f_{ij} = 1$, $q_{ij} = 1$ & $h_{ij} = 1$ if & only if $\bar{e}_{ij} \neq 0$, $\bar{f}_{ij} \neq 0$, $\bar{q}_{ij} \neq 0$ & $\bar{h}_{ij} \neq 0$

$e_{ij} = 0$, $f_{ij} = 0$, $q_{ij} = 0$ & $h_{ij} = 0$ otherwise

The accessibility matrix R may be determined graphically as follows:

$e_{ij} = 1$, $f_{ij} = 1$, $q_{ij} = 1$ and $h_{ij} = 1$ if there is a path between the peaks x_j, u_j, x_j, u_j and the peaks x_i, x_i, y_i, y_i respectively.

$e_{ij} = 0$, $f_{ij} = 0$, $q_{ij} = 0$ and $h_{ij} = 0$ otherwise.

Note that $h_{ij} = 1$ if $q_{i\ell} \cdot f_{\ell j} = 1$ for certain $\ell \in \{1,2,\ldots,n\}$.

Theorem (SIL-78)

System (C,A,B) is accessible from the input if and only if matrix F does not have a zero line, and has an accessible output if and only if matrix Q does not have a zero column and is accessible from the input-output if and only if matrix H has neither a a zero column, nor a zero line.

APPENDIX 4 /212

CALCULATION OF THE DERIVATIVES OF THE CRITERION USING THE VARIATIONS METHOD

To determine the derivatives of the criterion, we will use the same procedure as Levine and Athans (LEV-70). To accomplish this, the three following results are necessary:

Theorem (BEL-70) (SIN-81)

If the integral $X = -\int_0^\infty e^{At}\, C\, e^{Bt}$ exists for all C then it represents the unique solution of equation:

$$AX + XB = C$$

Theorem (BEL-70) (SIN-81)

For a small number ε, and by limiting ourselves to an expansion to the 1st order, we have:

$$e^{(A+\varepsilon B)t} = e^{At} + \varepsilon \int_0^t e^{A(t-s)} B e^{As} ds$$

Kleinman's Lemma (KLE-66)

If f(X) is a trace function. It may be expressed:

$$f(X+\varepsilon\Delta X) = f(X) + \varepsilon \text{ Trace } \{M(X) . \Delta X\}$$

therefore, when $\varepsilon \to 0$, we have:

$$\frac{df(X)}{dX} = M^T(X)$$

Let us consider the system:

$$\overset{o}{X} = AX + BU$$
$$Y = CX$$

and the quadratic criterion:

$$J = \int_0^\infty (X^T Q X + U^T R U) dt$$

Let us apply U = -KY, the closed loop system becomes:

$$\overset{o}{X} = (A-BKC) X - DX$$

and its solution is $X = e^{DT} X_0$

therefore the criterion becomes:

$$J(D) = \text{Tr} \{ \int_0^\infty e^{D^T t} Q_1(K,C) e^{Dt} dt X_o \}$$

with /213

$$X_o = E \{ X(o) X^T(o)\}$$
$$Q_1(K,C) = Q + C^T K^T R K C$$

Let us now assume that the system parameters undergo small variations, i.e.:

$$\left.\begin{array}{lll} A & \rightarrow & A + \varepsilon_A \cdot \Delta A \\ B & \rightarrow & B + \varepsilon_B \cdot \Delta B \\ C & \rightarrow & C + \varepsilon_C \cdot \Delta C \\ K & \rightarrow & K + \varepsilon_K \cdot \Delta K \end{array}\right|$$

with ε_A, ε_B, ε_C and ε_K the small real numbers of the same order of magnitude: $\varepsilon_A \cong \varepsilon_B \simeq \varepsilon_C \simeq \varepsilon_K$

The closed loop system is expressed:

$$\overset{o}{X} = (D + \varepsilon \cdot \Delta D)\, X$$

with $\Delta D = \Delta A - \Delta B\ KC - \Delta B . \Delta K . C - BK . \Delta C$

and the criterion becomes:

$$J(D+\varepsilon\Delta D) = \mathrm{Tr}\left[\int_0^{\infty} e^{(D+\varepsilon\Delta D)^T t}\, Q_1(K+\varepsilon\Delta K, C+\varepsilon\Delta C)\, e^{(D+\varepsilon\Delta D)t}\, dt\ X_o\right]$$

By expanding the above expression to the 1st order, taking into account the preceding theorems and properties as a function of the trace, we obtain:

$$J(D+\varepsilon\Delta D) = J(D) + \varepsilon\ \mathrm{Tr}\ \{2S(C^T K^T R - PB)\, \Delta(KC) + 2\ SP\ \Delta(A,B)\}$$

with:

$$\begin{array}{l} D^T P + PD + Q + C^T K^T RKC = 0 \\ DS + S\, D^T + X_o = 0 \\ \Delta D = \Delta A - \Delta B . KC - B . \Delta(KC) \\ \Delta(KC) = \Delta K . C + K .\ \Delta C \\ \Delta(A,B) = \Delta A - \Delta B . KC \end{array}$$

Two variation cases are considered:

1 - Variation in A and B only

In this case, we have:

from which

$$J\left[D+\varepsilon\Delta(A,B)\right] = J(D) + \varepsilon \, Tr \, \{2SP \; \Delta(A,B)\}$$

and the application of Kleinman's lemman gives:

$$\frac{\partial J}{\partial (A,B)} = 2 \; PS$$

this case combines the two situations: /214

.variation in A alone:

$$\frac{\partial J}{\partial A} = 2 \; PS$$

.Variation in B alone:

$$\frac{\partial J}{\partial B} = - \; 2 \; PS \; C^T \; K^T$$

2 - Variation in K and C alone

In this case we have:
$$\Delta D = - \; B \, \Delta(KC)$$
$$\Delta(A,B) = 0$$

and the criterion is expressed:

$$J\left[D - \varepsilon \; B \, \Delta(KC)\right] = J(D) + \varepsilon \, Tr \, \{(2S \; C^T K^T R \; - \; 2 \; SPB) \; \Delta(KC)\}$$

which gives us:

$$\frac{\partial J}{\partial (KC)} = 2 \; (RKC \; - \; B^T P) \; S$$

This formula combines two cases:

.variation in C alone

$$\frac{\partial J}{\partial C} = 2 \; K^T \; (RKC \; - \; B^T P) \; S$$

.variation in K alone:
$$\Delta D = \Delta(A,B)$$
$$\Delta(KC) = 0$$

APPENDIX 5

SYNTHESIS OF A DECENTRALIZED OBSERVER BY PARAMETRIC OPTIMIZATION /215

In our paper, we have discussed the synthesis of decentralized dynamic controls (decentralized observers). However this problem is the same as that of the synthesis of the decentralized static controls of an augmented system which we will determine now:

Given a system (with N control stations) with its global state equation:

$$\begin{aligned} \dot{X} &= A^*X = B^*U & \qquad & X \in R^n,\ U \in R^m \\ Y &= C^*X & & Y \in R^p \end{aligned} \qquad (A5.1)$$

Let us assume that the controls of subystems are created by dynamic compensators given by the following global equation:

$$\begin{aligned} \dot{Z} &= F\,Z + G\,Y & \qquad & Z \in R^q \\ U &= H\,Z + E\,Y & & \end{aligned} \qquad (A5.2)$$

where matrices F, G, H and E are constant of of the appropriate dimensions and verify the structural stresses on the control (block diagonal structure of a complete decentralization).

When control (A5.2) is applied to system (A5.1), the closed loop system is described by:

$$\dot{r} = \begin{bmatrix} \dot{X} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} A^* + B^* E C^* & B^* H \\ G C^* & F \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} = Dr \qquad (A5.3)$$

or again $\dot{r} = (A+BKC)\ r = Dr$ (A5.4)

with :

$$A = \begin{array}{cc} \left[\begin{array}{c|c} A^* & 0 \\ \hline 0 & 0 \end{array}\right] & \begin{array}{c} n \\ q \end{array} \\ \begin{array}{cc} n & q \end{array} & \end{array}, \quad B = \begin{array}{cc} \left[\begin{array}{c|c} B^* & 0 \\ \hline 0 & I_q \end{array}\right] & \begin{array}{c} n \\ q \end{array} \\ \begin{array}{cc} m & q \end{array} & \end{array} \quad K = \begin{array}{cc} \left[\begin{array}{c|c} E & H \\ \hline G & F \end{array}\right] & \begin{array}{c} m \\ q \end{array} \\ \begin{array}{cc} p & q \end{array} & \end{array} \quad C = \begin{array}{cc} \left[\begin{array}{c|c} C^* & 0 \\ \hline 0 & I_q \end{array}\right] & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} n & q \end{array} & \end{array}$$

The problem therefore consists of determining the matrix K (which represents the compensator) which stabilizes the closed loop system (A5.4) and minimizes the following quadratic criterion:

$$J = \int_o^\infty L(X,Z,U)\, dt$$

with: /216

$$L = \begin{bmatrix} X \\ Z \end{bmatrix}^T \begin{bmatrix} Q_X & Q_{Xz} \\ Q_{zX} & Q_z \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + U^T R^* U$$

$$L = r^T Q r + u^T R^* U$$

the control may be expressed:

$$U = H Z + E Y = (EC^* \quad H)\, r$$

$$= (E \quad H) \begin{bmatrix} C^* & 0 \\ 0 & I \end{bmatrix} r = (E \quad H)\, C\, r$$

We therefore have:

$$U^T R^* U = r^T C^T (E \quad H)^T R^* (E \quad H)\, C\, r$$

$$= r^T C^T \begin{bmatrix} E & H \\ G & F \end{bmatrix}^T \begin{bmatrix} R^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E & H \\ G & F \end{bmatrix} C\, r = r^T C^T K^T R K C r$$

Let us replace this relation in L, we obtain:

$$L = r^T (Q + C^T K^T R K C)\, r \quad = r^T Q_k r$$

The problem therefore takes on the following form:

$$\min_K J(K) = \int_o^\infty r^T Q_k r\, dt$$

$$\text{under} \quad \overset{o}{r} = (A+BKC)r = D\, r$$

This problem may be expressed (VIII.2)

$$\min_K J(K) = \text{Trace}\, (P\, r_o)$$

$$\text{under} \quad f(K) = D^T P + PD + Q_k = 0$$

$$F(K) = 0$$

$$\text{with} \quad r_o = r(o)\ r(o)^T$$

where F(K) represents the stress on the control structure.

The necessary optimality conditions of this problem are:

$$\frac{dJ}{dK} = 2\,(RKC - B^T P)\, S\, C^T$$

$$D^T P + PD + Q_k = 0$$

$$DS + S\, D^T + r_o = 0$$

and the solution may be obtained by one of the algorithms of VIII.2.

If the system parameters vary, the application of the algorithm developed in VIII.3 to the agumented system, for robust static controls, provides a robust observer for variations in the system parameters.

REFERENCES

AND-71 **ANDERSON** B.O.D., **MOORE** J., "Linear optimal control", Prentice Hall, 1971.

AND-81a **ANDERSON** B.O.D., **CLEMENTS** D.J., "Algebraic characterization of fixed modes in decentralized control", Automatica, vol. 17, n° 5, pp. 703-712, 1981.

AND-81b **ANDERSON** B.O.D., **MOORE** J., "Time-varying feedback laws for decentralized control", IEEE Trans. Auto. Contr. Vol. AC-26, n° 5, pp. 1133-1139, 1981.

AND-82 **ANDERSON** B.O.D., "Transfer function matrix description of decentralized fixed modes", IEEE Trans. Auto. Contr. Vol. AC-27, n° 6, pp. 1176-1182, 1982.

AND-84 **ANDERSON** B.O.D., **LINNEMANN** A., "Spreading the control complexity in decentralized control of interconnected systems", Systems & Control Letters, vol. 5, pp. 1-8, 1984.

AOK-68 **AOKI** M., "Control of large-scale dynamic systems by aggregation" IEEE Trans. Auto. Contr. vol. AC-13, pp. 246-253, 1968.

AOK-72 **AOKI** M., "On feedback stabilizability of decentralized dynamic systems", Automatica, vol. 8, n° 2, pp. 163-173, 1972.

AOK 73 **AOKI** M., **LI** M.T., "Controllability and stabilizability of decentralized dynamic systems", Proc. of Joint Automatic Control Conf. (JACC), Ohio State University (USA) pp. 278-186, 1973.

AOK-76 **AOKI M.**, "On decentralized stabilization and dynamic assignment problems", J. of Int. Economics, vol. 6, pp. 143-171, 1976.

ARM-81 **ARMENTANO V.A., SINGH M.G.**, "A new approach to the decentralized controller initialisation problem", The 8th World Congress of IFAC, paper 41.2, Kyoto (Japan), 1981.

ARM-82 **ARMENTANO V.A., SINGH M.G.**, "A procedure to eliminate decentralized fixed modes with reduced information exchange", IEEE Trans. Auto. Contr., vol. AC-27, n° 1, pp.258-260, 1982.

ATH-67 **ATHANS M., LEVINE W.S., LEVIS A.H.**, "A system for the optimal and suboptimal position and velocity control for a string of High-Speed Vehicles", Proc. of the 5th AICA Congress, Lausanne (Switzerland), 1967.

BAP-80 **BAPTISTELLA L.F.B.**, "Multi-criteria optimization of dynamic systems" PhD Thesis, University Paul Sabatier, Toulouse (France), 1980.

BAR-72 **BARTELS R.H., STEWART G.W**, "Solution of matrix equation AX + XB = C", Commun. ACM, vol. 15, n° 9, pp. 820, 1972.

BEL-70 **BELLMAN R.**, "Introduction to matrix analysis", Mc Graw-Hill, 2nd edition, 1970.

BEM-79 **BERTRAND P., SIRET J.M., MICHAILESCO G.**, "On the use of aggregation technics", in SINGH M.G. and A. TITLI (ed.) Handbook of large-scale systems engeneering application, North-Holland, Amsterdam (Netherlands), 1979.

BER-81 **BERNUSSOU J., GEROMEL J.C.**, "An easy way to find gradient matrix of composite matricial functions", IEEE Trans. Auto. Contr., Vol. AC-26, n° 2, pp. 538-540, 1981.

BIN-78 **BINGULAC S.P.**, "Calculation of derivatives of characteristic polynomials", IEEE Trans. Auto. Contr. vol. AC-23, n° 4, pp. 751-753, 1978.

BOW-76 **BOWIE W.S.**, "Application of graph theory in computer systems", Int. J. Comput. and Infor. Sci., vol. 5, pp. 9-31, 1976.

BRA-70 **BRASCH F.M., PEARSON J.B.**, "Pole placement using dynamic compensators" IEEE Trans. Auto. Control, vol. AC-15, pp. 34-43, 1970.

BUR-83 **BURROWS C.R., SAHINKAYA M.N.**, "A new algorithm for determining structural controllability". Int. J. Contr., vol. 33, n° 2, pp. 379-392, 1981.
"A modified algorithm for determining structural controllability", Int. J. Contr., vol. 37, n° 6, pp. 1417-1431, 1983.

CAL-78 **CALOVIC M., DJOROVIC M., SILJAK D.D.**, "Decentralized approach to Automatic Generation Control of interconnected power systems", Proc. of International Conference on Large High-Voltage Electric Systems (CIGRE), Paris (France), 1978.

CHE-70 **CHEN C.T.**, "Introduction to linear systems theory", Holt, Rinehart and Winston, New-York (USA), 1970.

CHE-78 **CHEMOUIL P.**, "Analysis and control of dynamic system with several time scales, PhD Thesis in Engineering, University of Nantes, 1978, (France).

CHE-81 **CHEN Y., SINGH M.G.**, "Certain practical consideration in model-following method of decentralized control", Proc. IEE, vol. 128, part D., n° 4, pp. 149-155, 1981.

CHE-84 **CHEN Y., MAHMOUD M.S., SINGH M.G.**, "An iterative block-diagonalization procedure for decentralized optimal control", Int. J. Systems Sci. Vol. 15, n° 5, pp. 563-573, 1984.

CHI-76 **CHIPMAN J.S.**, "Estimation and aggregation in econometric : application of the theory of generalized inverses", in Generalized Inverses and Applications, M.Z. NASLED Ed., New-York : Academic, 1976.

CHO-82 **CHONG C.Y.**, "Structural properties in decentralized control", 21th IEEE Conference on decision and control (CDC) pp. 1315-1316, 1982.

COR-76a **CORFMAT J.P., MORSE A.S.**, "Control of linear systems through specified input channels", SIAM J. Control and Optimization, vol. 14, n° 1, pp. 163-175, 1976.

COR-76b **CORFMAT J.P., MORSE A.S.**, "Decentralized control of linear multivariable systems", Automatica, vol. 12, n° 5, pp. 479-495, 1976.

DAV-74 **DAVISON E.J., WANG W.H.**, "Properties and calculation of transmission zeros of linear multivariable time-invariant systems", Automatica, vol. 10, pp. 643-658, 1974.

DAV-76a **DAVISON E.J.**, "Decentralized stabilization and regulation in large multivariable systems", In Direction in decentralized control, Many-Person optimization and Large Scale Systems (editors, Y.C. HO, S. MITTER), pp. 303-323, Plenum Press, 1976.

DAV-76b **DAVISON E.J.**, "The robust decentralized control of a general servomechanism problem", IEEE Trans. Auto. Contr., vol. AC-21, n° 1, pp. 14-24, 1976.

DAV-76c **DAVISON E.J., WANG W.H.**, "Remark on multiple transmission zeros of a system", Automatica, vol. 12, p. 195, 1976.

DAV-77 **DAVISON E.J.**, "Connectability and structural controllability of composite systems", Automatica vol. 13, pp. 109-123, 1977.

DAV-78a **DAVISON E.J., TRIPATHI N.**, "The optimal decentralized control of a large power system : load and frequency control", IEEE Trans. Auto. Contr., vol. AC-23, n° 2, pp. 312-325, 1978.

DAV-78b **DAVISON E.J., GESING W., WANG W.H.**, "An algorithm for obtaining the minimal realization of a linear time-invariant system and determining if a system is stabilizable - detectable", IEEE Trans. Auto. Contr., vol. AC-23, n° 6, pp. 1048-1054, 1978.

DAV-79 **DAVISON E.J.**, "The robust decentralized control of a servomechanism problem for composite systems with input-output interconnections", IEEE Trans. Auto. Contr., vol. 24, n° 2, pp. 325-327, 1979.

DAV-83 **DAVISON E.J., OZGUNER U.**, "Characterization of decentralized fixed modes for interconnected systems", Automatica, vol. 19, n° 2, pp. 169-182, 1983.

DAV-85a **DAVISON E.J., WANG S.H.**, "A characteirzation of decentralized fixed modes in terms of transmission zeros", IEEE Trans. Auto. Control, Vol. AC-30, n° 1, pp. 81-82, 1985.

DAV-85b **DAVISON E.J.** "Fixed modes and decentralized control", Encyclopedia of Systems and Control, Pergamon Press, to appear in 1985.

DAV-85c **DAVISON E.J.**, "Pole assignment", Encyclopedia of Systems and Control, Pergamon Press, to appear in 1985.

DEC-81 **DE CARLO R.A., SAEKS R.**, "Interconnected dynamical systems", Marcel Dekker Inc., 1981, New-York (USA).

DES-74 **DESOER C.A., SCHULMAN D.,** "Zeros and poles of matrix transfer function and their dynamical interpretation", IEEE Trans. Circ. Syst. vol. CAS-21, pp. 3-8, 1974.

EVA-84 **EVANS F.J., KRUSER M.,** "Pole assignment in decentralized systems : A structural approach", Proc. IEE, vol. 131, part D, n° 6, pp. 229-232, 1984.

FAD-63 **FADDEEV D.K., FADDEEVA V.N.,** "Computational methods of linear algebra" p. 288, Freeman 1963.

FEI-62 **FEINGOLD D.G., VARGA R.S.,** "Block diagonally dominant matrices and generalisation of the Gershgorin circle theorem", Pacific J. Math. vol. 12, pp. 1241-1250, 1962.

FES-79 **FESSAS P.S.** "A note on 'An example in decentralized control systems'", IEEE Trans. Auto. Contr., vol. AC-24, p. 669, 1979.

FES-80 **FESSAS P.S.,** "Decentralized control of linear dynamical systems via polynomial matrix methods" : "I - Two interconnected scalar systems", Int. J. Contr., vol. 30, n° 2, pp. 259-276, 197. "II Arbitrary interconnected systems", Int. J. Contr., vol. 32, n° 1, pp. 127-147, 1980.

FES-81 **FESSAS P.S.,** "Two equivalent approaches in decentralized control of linear systems", IEEE Trans. Auto. Contr., vol. AC-26, n° 6, pp. 1283-1285, 1981.

FES-82a **FESSAS P.S.,** "Matrix fraction description approach to decentralized control", Proc. IEE, Vol. 129, part D, n° 5, pp. 210-205, 1982.

FES-82b **FESSAS P.S.** "D-controllability of linear systems defined by their transfert matrices", Proc. IEE, vol. 129, part D, n° 5, pp. 206-210, 1982.

FOS-72 **FOSSARD A.,** "Control of multidimensional systems", Dunod 1972, Paris.

FOS-78 **FOSSARD A.,** "Representation of linear systems" E.N.S.A.E. 1978, Toulouse (France).

FOS-81 **FOSSARD A., MAGNI, J.F.,** "Modelization and control of systems with a multiple time scale, A.F.C.E.T. Congress 1981 Nantes (France).

FOS-84 **FOSSARD A.,** "Multi-time scale systems", Preprints of the 9th World congress of IFAC, Budapest (Hungary), July 1984.

FRA-79a **FRANKSEN O.I., FALSTER P., EVANS F.J.,** "Structural aspects of controllability and observability", part I and II", J. Franklin Inst., Vol. 308, pp. 79-124, 1979.

FRA-79b **FRANKSEN O.I., FALSTER P., EVANS F.J.,** "Qualitative aspects of large scale systems", Lecture Notes in Control and Information Sciences, vol. 17, Springer-Verlag 1979.

GAN-66 **GANTMACHER F.R.,** "Matrix theory", Dunod 1966, Paris (France).

GER-79a **GEROMEL J.C., BERNUSSOU J.,**"An algorithm for optimal decentralized regulation of linear quadratic interconnected systems", Automatica, vol. 14, pp. 489-491, 1979.

GER-79b **GEROMEL J.C., BERNUSSOU J.,** "Stability of two level control schemes subject to structural perturbations", Int. J. Contr., vol. 29, pp. 313-324, 1979.

GER-79c **GEROMEL J.C.,** "Study of dynamic interconnected systems: decentralization aspects, Phd Thesis.
Université Paul Sabatier, Toulouse (France), 1979.

GER-82 **GEROMEL J.C., BERNUSSOU J.**, "Optimal decentralized control of dynamic systems", Automatica, vol. 13, n° 5, pp. 545-557, 1982.

GER-84 **GEROMEL J.C., PERES P.L.D.**, "Decentralized load-frequency control", submitted for publication 1984.

GLO-76 **GLOVER K., SILVERMAN L.M.**, "Characterization of structural controllability", IEEE Trans. Auto. Contr., vol. AC-21, pp. 534-537, 1976.

GRU-80 **GRUMPOS P.P., LOPARO A.K.**, "Structural control of large scale systems" 19th IEEE Conference on Decision and Control, December 1980.

GRU-84 **GRUMPOS P.P.**, "Structural dynamic hierarchical stabilization and control of large scale systems", in "Control and Dynamic Systems : Advances in theory and application, 'Advances in decentralized distributed control and dynamic systems'", (Leondes C. ed.), Academic Press, New-York (USA), 1984.

GUA-83 **GUANGQUAN L., LEE G.K.F.**, "Decentralized control of large scale systems with dynamic interconnected subsystems", Int. J. Contr. vol. 37, pp. 995-1006, 1983.

HAR-65 **HARARY F., NORMAN R.Z., CARTWRIGHT D.**, "Structural models : An introduction to the theory of directed graphs", Wiley, New-York (USA), 1965.

HAS-78a **HASSAN M.F., SINGH M.G.**, "Robust decentralized controller for linear interconnected dynamical systems", Proc. IEE, vol. 125, n° 5, pp. 429-432, 1978.

HAS-78b **HASSAN M.F., SINGH M.G.**, "A hierarchical structure for computing near optimal decentralized control", IEEE Trans. Sys. Man & Cyb., vol. SMC-8, n° 7, pp. 575-579, 1978.

HAS-79 **HASSAN M.F., SINGH M.G., TITLI A.**, "Near optimal decentralized control with a pre-specified degree of stability", Automatica, vol. 15, pp. 483-488, 1979.

HAS-80 **HASSAN M.F., SINGH M.G.**, "Decentralized controller with online interaction trajectory improvement", Proc. IEE, vol. 127, part D, n° 3, pp. 142-148, 1980.

HOS-77 **HOSKINS W.D., MEEK D.S., WALTON D.J.**, "The numerical solution of $A^TQ + QA = - C$", IEEE Trans. Auto. Contr., vol. AC-15, n° 5, pp. 881-885, 1977.

HOS-80 **HOSOE S.**, "Determination of generic dimension of controllable subspace and its application", IEEE Trans. Auto. Contr., vol. AC-25, n° 6, pp. 192-1196, 1980.

HUJ-84 **HU Y.Z., JIANG W.S.**, "New characterization of decentralized fixed modes and their application", Preprint of the 9th Congress of I.F.A.C. Budapest (Hungary), July 1984.

IKE-79 **IKEDA M., SILJAK D.D.**, "Counter examples to Fessas conjecture", IEEE Trans. Auto. Contr., vol. AC-24, n° 4, p. 670, 1979.

IKE-81 **IKEDA M., SILJAK D.D., WHITE D.E.**, "Decentralized control with overlapping information sets" JOTA, vol. 34, n° 2, pp. 279-310, 1981.

IKE-84 **IKEDA M., SILJAK D.D.**, "An extension of the inclusion principle for overlapping decentralized control", Preprint of the 9th world congress of I.F.A.C., Budapest (Hungary), 1984.

ISA-73 **ISAKSEN L., PAYNE H.J.**, "Suboptimal control of linear systems by augmentation with application to Freeway Traffic Regulation", IEEE Trans. Auto. Contr., vol. AC-18, pp. 210-219, 1973.

JOH-84 **JOHNSTON R.D., BARTON G.W., BRISK M.L.**, "Determination of the generic rank of structural matrices", Int.J. Contr., vol. 40, n° 2, pp.

(original text cut off)

KAI-80 **KAILATH T.**, "Linear systems", Printic-Hall, 1980.

KAL-62 **KALMAN R.E.**, "Canonical structure of dynamical systems", Proc. Nat. Acad. Sci. U.S.A., vol. 48, pp. 596-600, 1962.

KAR-84 **KARMAKAR S.B.**, "An algorithm for finting a circuit of even length in a directed graph", Int.J. Sys. Sci., vol. 15, n° 11, pp. 1197-1201, 1984.

KAT-81 **KATTI S.K.**, "Comments on Decentralized control of linear multivariable systems", Automatica, vol. 17, n° 4, p. 665, 1981.

KAU-68 **KAUFMANN A.**, "Introduction to multivariable systems for applications", Dunod, Paris (France), 1978.

KEV-75 **KEVORKIAN A.K.**, "Structural aspects of large dynamic systems", Preprint of the 6th world congress of I.F.A.C., part III.A, paper 19.3, Boston (U.S.A.), 1975.

KLE-66 **KLEINMAN D.L.**, "On the linear regulator problem and the matrix Riccati equation", MIT Electronic Systems Lab., Cambridge (U.K.), Mass. Tech. Rept. ESL-R271, 1966.

KLE-68 **KLEINMAN D.L.**, "On the iterative technique for Riccati equation computation", IEEE Trans. Auto. Contr., vol. AC-13, pp. 114-115, 1968.

KLE-78 **KLEINMAN D.L.**, **RAO P.K.**, "Extension to the Bartels - Stewart algorithm for linear matrix equations" IEEE Trans. Auto. Contr., vol. AC-23, n° 1, pp. 85-87, 1978.

KOB-78 **KOBAYASHI H.**, **HANAFUSA H.**, **YOSHIKAWA T.**, "Controllability under decentarlized information structure", IEEE Trans. Auto. Contr., vol. AC-23, n° 2, pp. 182-188, 1978.

KOB-82 **KOBAYASHI H.**, **YOSHIKAWA T.**, "Graph-Theoretic approach to controllability and localizability of decentralized control", IEEE Trans. Auto. Contr., vol. AC-27, n° 5, pp. 1096-1108, 1982.

KOK-76 **KOKOTOVIC P.V.**, **O'MALLEY R.E.**, **SANNUTI P.**, "Singular perturbation and order reduction in control theory : An overview", Automatica, vol. 12, pp. 123-132, 1976.

KRO-67 **KROFT D.**, "All paths through a maze", Proc. IEEE, vol. 55, pp. 88-90, 1967.

KUL-82 **KU G.S.**, **LOPARO K.A.**, "Decentralized fixed modes and stabilization in large scale systems", Proc. of American Control Conference (ACC), Virginia (USA), pp. 874-877, June 1982.

LAN-74 **LANCASTER P.**, "On eigenvalues of matrices dependent on a parameter", Numerische Mathematik, vol. 6, pp. 377-387, 1964.

LEV-70 **LEVINE W.S.**, **ATHANS M.**, "On the determination of the optimal constant output feedback gains for linear multivariable systems", IEEE Trans. Auto. Contr. Vol. AC-15, n° 1, pp. 44-48, 1970.

LIN-74 **LIN C.T.**, "Structural controllability", IEEE Trans. Auto. Contr., vol. AC-19, n° 3, pp. 201-208, 1974.

LIN-83 **LINNEMANN A.**, "Fixed modes in parametrized systems", Int. J. Contr., vol. 38, n° 2, pp. 319-335, 1983.

LIN-84 **LINNEMANN A.**, "Decentralized control of dynamically Interconnected system', IEEE Trans. Auto. Contr., vol. AC-29, n° 11, pp. 1052-1054, 1984.

LOC-77 **LOCATELLI A., SCHIAVONI N., TARANTINI A.**, "Pole placement : role and choice of the underlying information pattern", Ricerche di Automatica, vol. 18, n° 1, pp. 107-126, 1977.

MAC-76 **MACFARLANE A.G.J., KARCANIAS N.**, "Poles and zeros of linear multivariable systems : a survey of the algebraic, geometric and complex-variable theory", Int. J. Contr., vol. 24, n°1, pp. 33-74, 1976.

MAH-80 **MAHALANABIS A.K., SINGH R.**, "On decentralized feedback stabilization of large scale interconnected systems", Int. J. Contr., vol. 32, n° 1, pp. 115-126, 1980.

MCB-72 **Mc BRINN D.E., ROY R.J.**, "Stabilization of linear multivariable systems by output feedback", IEEE Trans. Auto. Contr., vol. 17, pp. 243-245, 1972.

MEE-73 **MEERKOV S.M.**, "Vibrational control", Automation and Remote Control, vol. 31, pp. 201-209, 1973.

MEE-80 **MEERKOV S.M.**, "Principle of vibrational control : Theory and application", IEEE Trans. Auto. Contr., vol. AC-25, n° 4, pp. 755-762, 1980.

MIN-85 **MINZHI Z.**, "Comment on Decentralized control of Interconnected Dynamical Systems", IEEE Trans. Auto. Control, vol. AC-30, n° 3, pp. 319-320, 1985.

MOM-83 **MOMEN S., EVANS F.J.**, "Structurally fixed modes in decentralized systems, Part I : Two control stations, Part II : General case", Proc. IEE, vol. 130, part D, n° 6, pp. 313-327, 1983.

MOO-81 **MOORE B.C.**, "Principal component analysis in linear systems : Controllability, observability and model reduction", IEEE Trans. Auto. Contr., vol. AC-26, n° 1, pp. 17-32, 1981.

MOR-66 **MORGAN B.S.**, "Computational procedure for the sensitivity of an eigenvalue", Electronics Letters, vol. 26, pp. 197-198, 1966.

MOR-73 **MORSE A.S.**, "Structural invariants of linear multivariable systems", SIAM J. Control, vol. 11, n° 3, pp. 446-465, 1973.

MOR-78 **MORARI M., STEPHANOPOULOS G.**, "Comment on finding the generic rank of a structural matrix", IEEE Trans. Auto. Contr., vol. AC-23, n° 3, pp. 509-510, 1978.

MOR-82 **MORTAZAVIAN H.**, "On k-controllability and k-observability of linear systems", Proc. of the 5th International Conference on Analysis and Optimization of Systems", Versailles (France), december 1982.

OZG-83 **OZGUNER U., LEE L.C., DAVISON E.J.**, "Minimal realisation of a class of decentralized systems", Proc. of American Control Conference (ACC), San Francicso (U.S.A.), June 22-24, 1983.

PAR-74 **PARASKEVOPOULOS P.N., TSONIS C.A., TZAFESTAS S.G.**, "Eigenvalue sensitivity of linear time-invariant control systems with repeated eigenvalues", IEEE Trans. Auto. Contr., vol. AC-19, pp. 610-612, 1974.

PET-84 **PETEL R.V., MISRA P.**, "A numerical test for transmission zeros with application in characterizing decentralized fixed modes", Proc. of 23rd Conference on Decision and Control (C.D.C.), Las Vegas (U.S.A.), dec. 1984.

PIA-79 **PIASCO J.M., DIEP D.**, "Numerical automization of the water-steam cycle of ships, appendix 1; Description of the process, of Modelization and identification. Final report of DGRST/ENSM/IRCN Contract 76.7.1394, managed by R. Mezencev, December 1979.

PIC-83a **PICHAI V., SEZER M.E., SILJAK D.D.**, "A graphical test for structurally fixed modes", Mathematical Modelling, vol. 4, pp. 339-348, 1983.

PIC-83b **PICHAI V., SEZER M.E., SILJAK D.D.**, "A graph theoretic algorithm for hierarchical decomposition of dynamic systems with application to estimation and control", IEEE Trans. Syst. Man. & Cyb., vol. SMC-13, n° 3, pp. 197-207, 1983.

PIC-84 **PICHAI V., SEZER M., SILJAK D.D.**, "A graph-theoretic characterization of structurally fixed modes", Automatica, vol. 20, n° 2, pp. 247-250, 1984.

POT-79 **POTTER J.M., ANDERSON B.O., MORSE A.S.**, "Single-channel control of a two-channel system", IEEE Trans. Auto. Contr., vol. AC-24, pp. 491-492, 1979.

PUR-82 **PURVIANCE J.E., TYLEE J.L.**, "Scalar sinusoïdal feedback laws in decentralized control", Proc. of 21th IEEE Conference on Decision and Control, Florida (USA), December 1982.

RAM-82 **RAMAKRISHNA A., VISWANADHAM N.**, "Decentralized control of Interconnected dynamical systems", IEEE Trans. Auto. Contr., vol. AC-27, n° 1, pp. 159-164, 1982.

REI-81 **REINSCHKE K.J.**, "Structurally complete systems with minimal input and output vectors", Large scale systems, vol. 2, pp. 235-242, 1981.

REI-83 **REINSHKE K.J.**, "Graph-theoretic approach to control system", Proc. Thrid Conference on System Science, Lerchendal (Norway), Oct. 1983.

REI-84a **REINSCHKE K.J.**, "Graph-theoretic characterization of fixed modes in centralized and decentralized control", Int. J. Contr., vol. 39, n° 4, pp. 715-729, 1984.

REI-84b **REINSCHKE K.J.**, "Graph-theoretic characterization of structural properties of paths and cycle families", Preprint of the 9th world congress of I.F.A.C., Budapest (Hungary), July 1984.

ROS-65a **ROSENBROCK H.H.**, "Transfer matrix of linear dynamic system", Electronics Letters, vol. 1, n° 4, pp. 95-96, 1965.

ROS-65b **ROSENBROCK H.H.**, "Sensitivity of an eigenvalue to changes in the matrix", Electronics Letters, Vol. 1, n°10, pp. 278-279, 1965.

ROS-70 **ROSENBROCK H.H.**, "State space and multivariable theory", Nelson (London), 1970.

ROS-74 **ROSENBROCK, H.H.**, "The zeros of a system", Int.J. Contr., vol. 18, n° 2, pp. 297-299, 1973. "Correction to 'The zeros of a system'", Int. J. Contr., vol. 20, p. 525, 1974.

ROY-70 **ROY B.**, "Modern algebra and graph theory", Dunod, Paris, (France) 1970.

SAE-79 **SAEKS R.**, "On the decentralized control of interconnected dynamical systems", IEEE Trans. Auto. Contr., vol. AC-24, n° 2, pp. 269-271, 1979.

SAN-78 **SANDELL N.R., VARAIYA P., ATHANS M., SAFANOV M.G.**, "Survey of decentralized control methods for large scale systems", IEEE Trans. Auto. Contr., vol. AC-23, n° 2, pp. 108-128, 1978.

SEN-79 **SENNING M.F.**, "Feasibly decentralized control", Ph. D. THesis ETH Zurich (Switzerland), 1979.

SER-82 **SERAJI H.**, "On fixed modes in decentralized control systems", Int. J. Contr., vol. 35, n° 5, pp. 775-784, 1982.

SEZ-78 **SEZER M.E., HUSEYIN O.**, "Stabilization of linear time-invariant interconnected systems using local state feedback", IEEE Trans. Sys. Man. & Cyb., vol. SMC-8, pp. 751-756, 1978.

SEZ-80 **SEZER M.E., HUSEYIN O.**, "On decentralized stabilization of interconnected systems", Automatica, vol. 16, pp. 205-209, 1980.

SEZ-81a **SEZER M.E., SILJAK D.D.**, "On structurally fixed modes", Proc. of IEEE Inernational Symposium on Circuit and Systems, pp. 558-565, Chicago (USA), 1981.

SEZ-81b **SEZER M.E., SILJAK D.D.**, "Structurally fixed modes", Systems & Control Letters, vol. 1, n° 1, pp. 60-64, 1981.

SEZ-81c **SEZER M.E., HUSEYIN O.**, "Comment on decentralized state feedback stabilization", IEEE Trans. Auto. Contr., vol. AC-26, n° 2, pp. 547-549, 1981.

SEZ-83 **SEZER M.E.**, "Minimal essential feedback patterns for pole assignment using dynamic compensation", Proc. of 2th IEEE Conference Decision and Control, December 1983.

SHI-76 **SHIELDS R.W., PEARSON J.B.**, "Structural controllability of multi-input linear systems", IEEE Trans. Auto. Contr., vol. AC-21, n° 2, pp. 203-212, 1976.

SIL-79 **SILJAK D.D.**, "Nonlinear systems", Wiley, New-York (USA), 1969.

SIL-76a **SILJAK D.D.**, "Multilevel stabilization of large scale systems : A spining flexible spacecraft", Automatica vol. 12, pp. 309-320, 1976.

SIL-76b **SILJAK D.D., SUNDARESHAN S.K.**, "A multilevel optimization of large scale dynamic systems", IEE Trans. AUto. Contr., vol. AC-21, pp. 79-84, 1976.

SIL-78 **SILJAK D.D.**, "Large scale systems : stability and structure", North-Holland, New-York (USA), 1978.

SIL-82a **SILJAK D.D. and al**, "The inclusion principle for dynamic systems", Final report DE-AC037 ET 29138-34, Santa Clara University (USA), 1982.

SIL-82b **SILJAK D.D., PICHAI V., SEZER M.E.**, "Graph-theoretic analysis of dynamic systems", Report DE-AC037 ET 29138-35, Santa Clara University (USA), 1982.

SIL-83 **SILJAK D.D.**, "Complex dynamic systems : Dimensionality, structure and uncertainty", Large Scale System, vol. 4, pp. 279-294, 1983.

SIM-62 **SIMON H.A.**, "The architecture of complexity", Proc. Of American Philosophical Society, vol. 106, pp. 467-482, 1962.

SIN-78a **SINGH M.G., TITLI A.**, "Systems : Decomposition, Optimization and control", Pergamon Press, London (U.K.), 1978.

SIN-78b **SINGH M.G.**, "Dynamical hierarchical control", North-Holland, Amsterdam (Netherlands), 1978.

SIN-81 **SINGH M.G.**, "Decentralized control", North-Holland, Amsterdam (Nether-Lands), 1981.

SIN-83 **SINGH M.G., TITLI A., MALINOWSKI K.**, "Decentralized decision making and control : An overview", in STRASZAK A. : Large scale systems theory and application, Proc. of the I.F.A.C./I.F.O.R.S. Symposium on Large scale systems, Warsaw (Poland), Pergamon Press, Oxford (U.K.). 1983.

SOL-81 **SOLIMAN H.M., DARWISH M., FANTIN J.**, "Decentralized and hierarchical stabilization techniques for interconnected power systems", Large scale systems, vol. 2, pp. 113-122, 1981.

SOL-85 **SOLIMAN H.M.**, "Decentralized stabilization", Encyclopedia of Systems and Control, Pergamon Press, to appear in 1985.

SRI-79 **SRIMANI P.K., SENGUPTA A.**, "Algebraic determination of circuits in a directed graph", Int. J. Sys. Sci., vol. 10, pp. 1409-1413.

SUN-77 **SUNDARESHAN K.**, "Exponential stabilization of large scale systems. Decentralized and multilevel schemes", IEEE Trans. Sys. Man. & Cyb., vol. SMC-7, pp. 478-483, 1977.

TAR-84 **TAROKI M.**, "Fixed modes in decentralized control systems", Preprint of the first European Workshop on the "Real time control of large scale systems", University of Patras (Greece), July 1984.

TAR-84a **TARRAS A.M., TITLI A.**, "On a new algebraic characterization of decentralized fixed modes", Preprint of IEEE International Conference on Computers, Systems & Signal Processing, Bangalore (India), december 1984.

TAR-84b **TARRAS A.M., TITLI A.**, "Sensitivity characterization of decentralized fixed modes and its application", en cours de soumission à la revue IEEE Trans. Auto. Control.

TAR-85a **TARRAS A.M., TITLI A.**, "Sufficient reduced information pattern to avoid structurally fixed modes", Rapport interne L.A.A.S..

TAR-85b **TARRAS A.M., TITLI A.**, "ON the design of robust decentralized controllers" en cours de soumission à "IFAC Workshop on Automatic Control in Petroleum, Petrochemical and Desalination Industries", Kuwait, Nov. 18-20, 1985.

TAR-85c **TARRAS A.M., TRAVE L., TITLI A.**, "Minimal feedback structure avoiding structurally fixed modes", Rapport Interne L.A.A.S., en cours de soumission à la revue "International Journal of Control", 1985.

TAR-85d **TARRAS A.M., TITLI A.**, "On decentralized stabilization in presence of non structurally fixed modes", en cours de soumission à la revue IEEE Trans. Auto. Control, 1985.

TAR-86 **TARRAS A.M., TITLI A.**, "Sensitivity approach to decentralized control systems", en cours de soumission à "Special Issue on Large-Scale Systems" de la revue "Control : Theory and Advanced Technology (C-TAT)". A paraître en 1986.

THA-60 **THALER G.T., BROWN R.G.**, "Analysis and design of feedback control systems", Mc Graw-Hill, New-York (U.S.A.), 1960.

THI-71 **THIRIEZ M.**, "The set covering problem : a group theoretic approach", R.A.I.R.O., vol. 3, pp. 84-103, 1971.

TIT-79 **TITLI A.**, "Partitioning and time-decomposition for the control of interconnected systems", Proc. of the I.F.A.C. Symposium on "Optimization methods application aspects", Varna (Bulgaria), Pergamon Press (U.K.), 1979.

TIT-83a **TITLI A.**, "Complex systems methodology : An overview", First International workshop on methodology and applications of complex systems theory, Cairo (Egypt), November 1983.

TIT-83b **TITLI A., BERNUSSOU J., TRAVE L., TARRAS A.M.**, "State of art report on "Decentralized control", edited by M.G. SINGH, Control System Centre, UMIST, Manchester (U.K.), E.E.C. Contract n° 003277, 1983.

TIT-86 **TITLI A., TRAVE L., TARRAS A.M.**, "Large Scale Systems : Decentralization, structure constraints and fixed modes", Study now being submitted to Plennum Press, New-York (U.S.A.).

TRA-84a **TRAVE L., TARRAS A.M., TITLI A.**, "Some problems in decentralized control in presence of fixed modes", Preprints of the 9th world congress of I.F.A.C., Budapest (Hungary), July 1984.

TRA-84b **TRAVE L.**, "Output feedback control with decentralized structure : Notion of fixed modes" Phd thesis in Engineering, I.N.S.A. Toulouse (France), 1984.

TRA-85a **TRAVE L., TARRAS A.M., TITLI A.**, "An application of vibrational control to cancel unstable decentralized fixed modes", IEEE Trans. Auto. Contr., vol. AC-30, n° 3, pp. 283-286, 1985.

TRA-85b **TRAVE L., TARRAS A.M., TITLI A.**, "A graph theoretic algorithm for pole assignment with minimal cost feedback structure", rapport interne L.A.A.S. 1985, en cours de soumission à la revue "Systems and Control Letters".

VAN-68 **VAN TRESS H.L.**, "Detection, Estimation and Modulation theory", Part I, John Wiley & Sons, New-York (U.S.A.), 1968.

VID-82 **VIDYASAGAR M., VISWANADHAM N.**, "Algebraic characterization of decentralized fixed modes and pole assignment", Proc. of 21th IEEE Conference on Decision and Control, pp. 501-505, 1982.

VID-83 **VIDYASAGAR M., VISWANADHAM N.**, "Algebraic characterization of decentralized fixed modes", Systems & Control Letters, vol. 3, pp. 69-73, 1983.

WAN-73a **WANG S.H., DAVISON E.D.**, "Properties of linear time-invariant multivariable system subject to arbitrary output and state feedback", IEEE Trans. Auto. Contr., vol. AC-18, pp. 24-32, 1973.

WAN-73b **WANG S.H., DAVISON E.D.**, "On the stabilization of decentralized control systems", IEEE Trans. AUto. Contr., vol. AC-18, pp. 437-478, 1973.

WAN-78a **WANG S.H., DAVISON E.D.**, "Minimization of transmission cost in decentralized control systems", Int. J. Contr., vol. 28, n° 6, pp. 889-896, 1978.

WAN-78b **WANG S.H.**, "An example in decentralized control", IEEE Trans. Auto. Contr., vol. AC-23, n° 5, p. 938, 1978.

WAN-82 **WANG S.H.**, "Stabilization of decentralized control systems via time-varying controllers", IEEE Trans. Auto. Contr., vol. AC-27, n° 3, pp. 741-744, 1982.

WIL-85 **WILLEMS J.L.**, "Decentralized stabiliation", Encyclopedia of Systems and Control, Pergamon Press, to appear in 1985.

WOL-73a **WOLOVICH W.A.**, "On determining the zeros of state-space systems", IEEE Trans. AUto. Contr., vol. AC-18, pp. 542-544, 1973.

WOL-73b **WOLOVICH W.A.**, "On the numerators and zeros of rational transfer matrices", IEEE Trans. Auto. Contr., vol. AC-18, pp. 544-546, 1973.

WON-67 **WONHAM W.M.**, "On pole assignment in multi-input controllable linear systems", IEEE Trans. Auto. Contr., vol. AC-12, pp. 660-665, 1967.

WON-74 **WONHAM W.M.**, "Linear multivariable control : A geometric approach", Springer-Verlag (Berlin), 1974.

XIN-82 **XINOGALAS T.C., MAHMOUD M.S., SINGH M.G.**, "Hierarchical computation of decentralized gain for interconnected systems", Automatica, vol. 18, n° 4, pp. 473-478, 1982.

YAH-77 **YAHAGI T.**, "Optimal output feedback control with reduced performance index sensitivity", Int. J. Contr., vol. 25, n° 5, pp. 769-783, 1977.

ZHE-84 **ZHENG Y.F.**, "The study of local controllability and observability of decentralized systems via polynomial models', Preprint 84-002 of Department of Mathematics - East China Normal University, Shanghai (China), 1984.

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16. Abstract

This dissertation thesis deals with the problem of stabilization/pole placement under structural constraints of large scale linear systems. The existence of a solution to this problem is expressed in terms of FIXED MODES (ch. II). The aim of this work is to provide a bibliographic survey of the available results concerning the fixed modes (Characterization, Elimination, Control structure selection to avoid them, Control design in their absence) and to present our contribution to this problem which can be summarized by the use of mode sensitivity concept to detect (ch. III - IV) or to avoid (ch. VII) them, the use of vibrational control to stabilize them (ch. VI), and the addition of parametric robustness considerations to design an optimal decentralized robust control.

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